ELEMENTS

OF

ANALYTICAL GEOMETRY.

ELEMENTS

OF

ANALYTICAL GEOMETRY

AND OF THE

DIFFERENTIAL AND INTEGRAL

CALCULUS.

BY

CHARLES DAVIES, LL.D.,

PROFESSOR OF HIGHER MATHEMATICS, COLUMBIA COLLEGE.

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PREFACE.

THE method, invented by DESCARTES, of representing all the parts of a geometrical magnitude by a single equation, has wrought an entire change in the mathematical and physical sciences.

Before his time, the *Measures* of the Geometrical Magnitudes were alone subjected to the processes of Algebra. He extended the analysis to the determination of their positions and forms. By this happy invention, modes of investigation at once difficult and disconnected, and dependent for success, in each particular case, on the skill and ingenuity of the inquirer, and often on accident, are reduced to a simple and uniform process. The great work of LA PLACE, is the legitimate fruit of this discovery. Here, all the formulas necessary for determining the positions and motions of the bodies of the solar system, are deduced from the single law of gravitation, and expressed in general equations.

The first edition of the Analytical Geometry was published in 1836. It was designed, specially, for the pupils of the Military Academy, and in its construction, little

attention was paid to the supposed wants of other Institutions. It is now presented to the public under a different form, and designed to fill a very different place. In 1836, the study of Mathematics, and especially in the higher branches, was limited, comparatively, to a few Institutions of the higher grade. Now, it is pursued in all our Colleges, and in many of our Academies and High Schools. It forms a part, and indeed an important part, of our system of Public Instruction.

To prepare a work that shall cultivate this growing taste for mathematical science, in one of its most attractive departments—not too large for general use, and yet containing all the great principles, rightly arranged and properly discussed—has been attended with some difficulties. How far they have been overcome, the public will judge. The present work is supposed to contain all that is necessary to the general student. The Table of Contents indicates the subjects, and the order in which they are treated.

COLUMBIA COLLEGE, *New York*, 1860.

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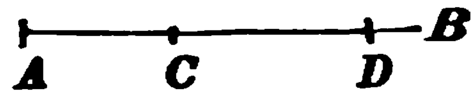
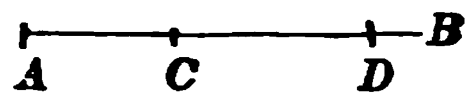
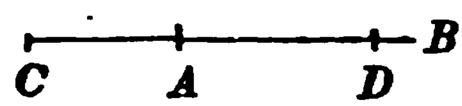


Draw an indefinite right line AB. From any point as A, lay off a distance AC equal to a, and then from C, a distance CD equal to b, and AD will be equivalent to $a + b$.

4. Construct the expression, $a - b$.



Draw an indefinite right line AB. From any point as A, lay off a distance AD equal to a, and then from D, a distance DC, *in the direction towards* A, equal to b; AC will then express the difference between a and b, and hence, is equivalent to $a - b$.

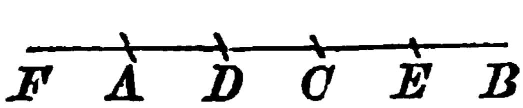


If b is greater than a, $a - b$ will be essentially negative: AC will then be negative, which is shown by the point C falling at the left of A.

We have, in this example, the geometrical interpretation of the negative sign: viz.

*If distances in one direction are regarded as positive, those in a contrary direction must be regarded as negative.**

5. Construct the expression, $a - b + c - d$.



Draw an indefinite line AB. From any point, as A, lay off the distance AC equal to a, and then the distance CD, in the opposite direction, equal to

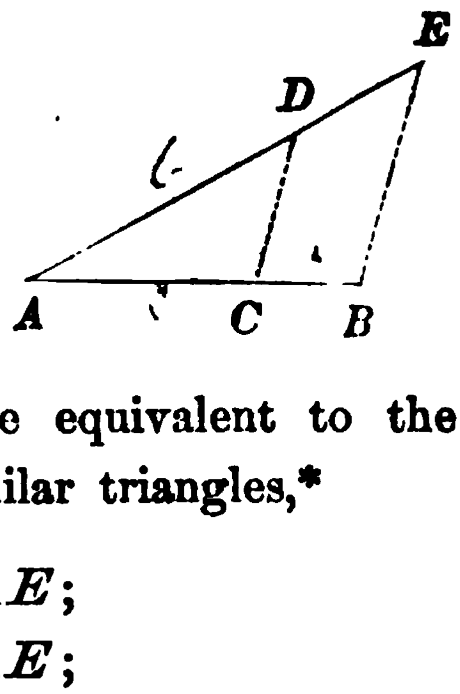
* Bourdon, Art. 89. University, Art. 96.

NOTE.—All the references are to Davies' Bourdon, Davies' University Algebra, and Davies' Legendre, Geometry, and Analytical Trigonometry.

b. From D, lay off DE, to the right, equal to c, and then from E, lay off EF, to the left, equal to d. The line AF will be equivalent to the *algebraic* expression. It will be negative, because the sum of the negative terms, in the algebraic expression, exceeds the sum of the positive terms, and this is indicated by the direction of the line AF. From the above examples, we conclude that,

Every algebraic expression of the first degree will represent a line; whence, it is called, *linear.*

6. Construct the expression, $\dfrac{ab}{c}$



Draw two indefinite right lines AB, AE, making any angle with each other. From A, lay off a distance $AC = c$, also the distance $AB = a$. Then from A, lay off $AD = b$; join C and D, and through B draw BE parallel to CD; then will AE be equivalent to the given expression. For, we have by similar triangles,*

$$AC : AB :: AD : AE;$$

that is, $$c : a :: b : AE;$$

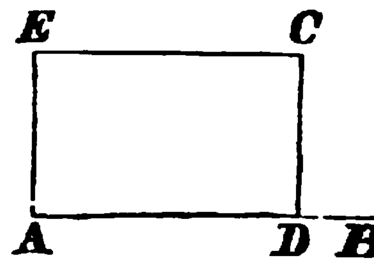
therefore, $$AE = \frac{ab}{c}.$$

Construction of expressions of the second degree.

7. Construct the expression, ab.

The degree of a term is the number of its literal factors.† Hence, ab is of the second degree.

* Legendre, Bk. IV. Prop. 15. † Bourdon, Art. 25. Univ. Art. 12.

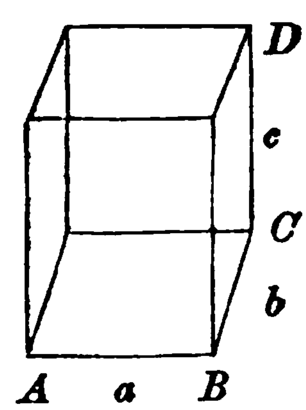


Draw the indefinite straight line AB. Lay off, from A to D, as many units of length as there are units in a. At D, draw DC perpendicular to AB, and make it equal to as many units of length as there are units in b. Then, the rectangle $ADCE$, will contain as many units of surface as there are units in the expression, $a \times b$. Hence, we conclude that,

*Every algebraic expression of the second degree represents a surface.**

Construction of an expression of the third degree.

8. Construct the expression abc.



Draw an indefinite line, and lay off AB equal to the number of units in a. Draw, in the plane of the paper, BC, perpendicular to AB, and make it equal to b. At C, suppose CD to be drawn perpendicular to the plane of the paper, and made equal to c. Then, having drawn the other lines of the figure, ABC-D will be a rectangular parallelopipedon equivalent to the expression abc. Hence,

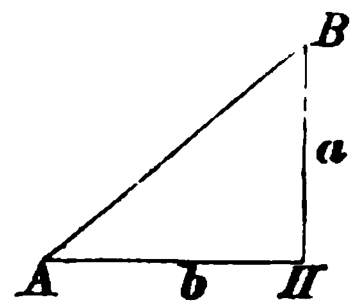
Every algebraic expression of the third degree will represent a volume.†

Construction of expressions of the zero degree.

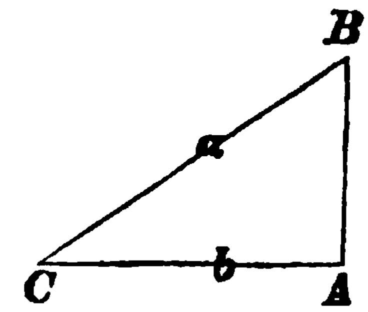
9. Construct the expression, $\frac{a}{b}$.

* Legendre, Bk. IV. Prop. 4. Sch. 1. † Bk. VII. Prop. 13. Sch.

Since there is one literal factor in the numerator, and one in the denominator, the quotient is an abstract number; and hence, contains no literal factor: therefore, the degree of the term is 0. Draw AH, and make it equal to b. At H, draw HB, perpendicular to AH, and make it equal to a, and join A and B. Then,



$\frac{a}{b}$ = tan A to the radius 1.*



If a were made the hypothenuse, $\frac{b}{a}$ would denote the cosine C, or sin B to the radius 1. Hence,

Every algebraic expression of the zero degree represents the sine, cosine, tangent, &c., of an arc or angle, to the radius 1.

It follows, therefore, that every abstract number has a geometrical interpretation; for, *it will always denote some function of an arc described with the radius* 1.

Homogeneity of terms.

10. We have seen, that there are four kinds of algebraic terms, which may represent geometrical magnitudes; viz. terms of the 1st degree, which represent lines; terms of the 2d degree, which represent surfaces; terms of the third degree, which represent volumes; and terms of the zero degree, which represent the functions of angles to the radius 1.

Since no other magnitudes occur in geometry, no alge-

* Legendre, Trig. Arts. 30–31.

braic term of a higher degree than the third, can have a geometrical equivalent. If such a term occur, we can only find its geometrical equivalent by regarding all the factors but three, as numerical.

Since like quantities, only, can be added or subtracted, it follows, that if two or more terms are connected together by the signs + or −, they must be homogeneous.* If they are not so, in form, it is because the geometrical unit of length, generally denoted by 1, has been omitted in the algebraic expressions, wherever it has occurred as a factor or a divisor; and this must be restored before finding the geometrical equivalent. Thus, if we have,

$$ab + c,$$

the first term is of the second degree, and the second, of the first degree. The degree of the second term is changed (without altering its numerical value), by introducing the linear unit 1, as a factor, and we then have,

$$ab + 1 \times c,$$

which is a homogeneous expression.

If we have the expression,

$$a + bc - dfg,$$

it may be made homogeneous by introducing the factors of 1; we then have,

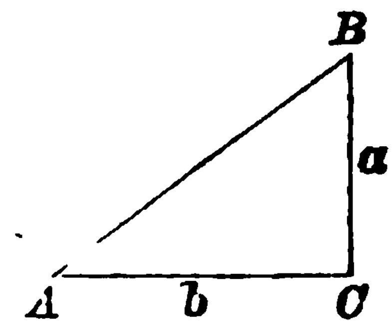
$$1 \times 1 \times a + 1 \times bc - dfg;$$

which is homogeneous.

* Bourdon, Art. 26. University, Art. 12.

EXAMPLES IN CONSTRUCTION.

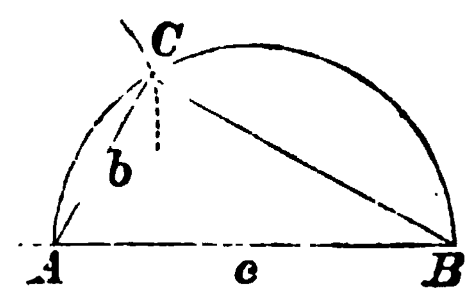
1. Construct the expression, $a^2 + b^2$.



Draw an indefinite right line, and lay off $AC = b$. At C, draw CB perpendicular to AC, and make it equal to a, and draw AB: then, $\overline{AB}^2$ will be the equivalent of

$$a^2 + b^2*.$$

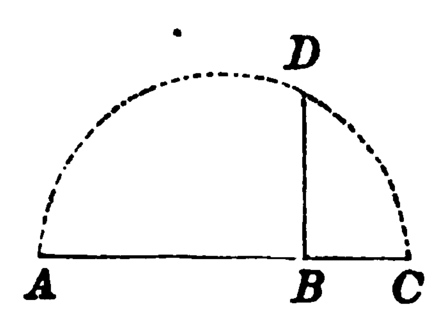
2. Construct the expression, $c^2 - b^2$.



Draw an indefinite line, and on it lay off $AB = c$; and on AB, as a diameter, describe a semicircumference ACB. With A as a centre, and a radius equal to b, describe an arc, intersecting the circumference at C. Then draw AC and CB.

Now, since ACB is a right-angled triangle, $\overline{AB}^2 - \overline{AC}^2$ is equivalent to $\overline{CB}^2$:† hence, $\overline{CB}^2$ is the equivalent of $c^2 - b^2$.

3. Construct the expression, $\sqrt{ab}$, and $\sqrt{a}$.



Draw an indefinite right line ABC, and from any point, as A, make $AB = a$, and then $BC = b$. On AC, as a diameter, describe a semi-circumference, and from B, draw BD perpendicular to AC, intersecting the circumference at D: then, BD will be the equivalent magnitude.

* Leg., Bk. IV. Prop. 11. † Leg., Bk. IV. Prop. 11, Cor. 1.

For, $$\overline{BD}^2 = AB \times BC* = a \times b;$$

hence, by extracting the square root of both members,

$$BD = \sqrt{ab}.$$

If we have $\sqrt{a}$, we have simply to introduce, under the radical, the factor 1, thus making the expression of the second degree.

We then have, $\sqrt{a} = \sqrt{a \times 1}.$

Making $AB = a$, and $BC = 1$, we have the same construction as before.

4. Construct the roots of the equation of the first form,

$$x^2 + 2px = q.†$$

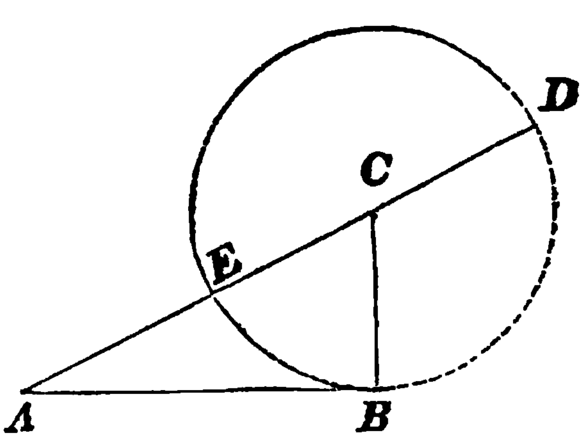
After making the second member of the equation homogeneous, and placing it equal to b^2, we have,

$$x^2 + 2px = 1 \times q = b^2.$$

This equation can be put under the form,

$$x(x + 2p) = b^2;$$

from which we see, that b is a mean proportional between x and $x + 2p$.



To construct these values of x, draw AB, and make it equal to b. At B, erect the perpendicular BC, and make it equal to p, and join A and C. With C as a centre, and CB as a radius, describe

* Leg., Bk. IV. Prop. 23, Cor. † Bour., Art. 117. Univ., Art. 147.

a semi-circumference cutting AC in E, and AC produced, in D; then will AE be equal to x. For,*

$$AE(AE + 2EC) = \overline{AB}^2 = b^2;$$

or, $$x(x + 2p) = b^2.$$

Finding the roots of the given equation, we have,

$$x' = -p + \sqrt{b^2 + p^2}, \text{ and } x'' = -p - \sqrt{b^2 + p^2}.$$

Having constructed the triangle ABC, as before, AC will represent the radical part of the values of x.

For the first value of x, the radical is positive, and is laid off from A towards C: then $-p$ is laid off from C to E, leaving AE positive, as it should be, since it is estimated from A towards C.

For the second value of x, we begin at D, and lay off DC equal to $-p$; we then lay off the minus radical from C to A, giving $-DA$, for the second value of x.

$- AD$

Let us now see if this value of x will satisfy the equation,

$$-x(-x + 2p) = b^2,$$

or, $$-AD(-AE) = b^2,$$

or, $$AD \times AE = \overline{AB}^2.$$

Hence, the two values of x, are $+AE$, estimated from A towards D, and $-DA$, estimated from D towards A.

5. Construct the roots of the equation of the second form,

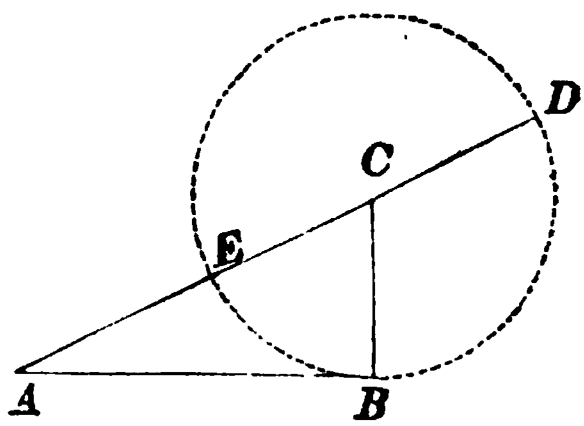
$$x^2 - 2px = q = 1 \times q = b^2.$$

* Legendre, Bk. IV. Prop. 30.

Finding the roots of this equation, we have,

$$x' = p + \sqrt{b^2 + p^2}, \quad \text{and} \quad x'' = p - \sqrt{b^2 + p^2}.$$

To construct these values of x. Having constructed the figure, as in the last example, the first value of x will be the line AD, estimated from A to D.



The second value will be $+ EC - CA$, the first estimated from E to C, and the latter from C to A: this leaves, for the reduced value $- EA$, estimated from E to A.

The positive root, in the construction for the first form, corresponds to the negative root in the construction for the second; and the negative root in the first, to the positive root in the second. This is as it should be, since either of the forms changes to the other, by substituting $- x$ for x.

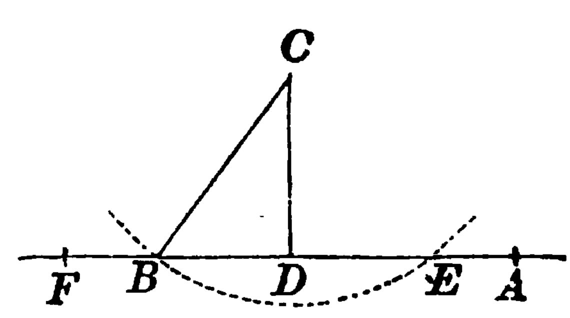
6. Construct the roots of the equation of the third form.

$$x^2 + 2px = - q = - 1 \times q = - b^2.$$

Solving the equation, we have,

$$x' = - p + \sqrt{p^2 - b^2}, \quad \text{and} \quad x'' = - p - \sqrt{p^2 - b^2}.$$

To construct these values, draw an indefinite right line FA, and from any point, as A, lay off a distance $AD = -p$, and, since p is negative, we lay off its value to the left. At D, draw DC perpendicular to



FA and make it equal to b. With C as a centre, and $CB = p$ as a radius, describe the arc of a circle cutting FA, in B and E. Now, the value of the radical quantity will be BD or DE. The first value of x will be $-AD$ plus DE, equal to $-AE$. The second, will be $-AD +(-DB)$ equal to $-AB$: so that both of the roots, being negative, are estimated in the same direction from A, to the left.

Therefore, the two roots are $-AE$, and $-AB$.

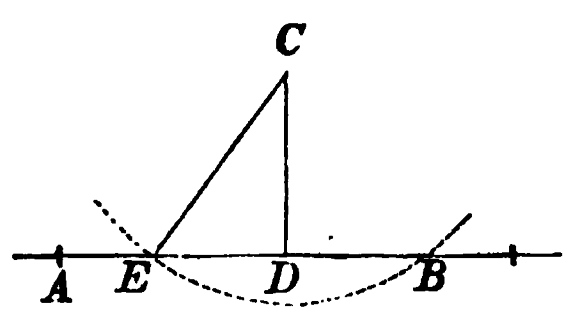
7. Construct the roots of the equation of the fourth form,

$$x^2 - 2px = -q = -1 \times q = -b^2.$$

Solving the equation, we have,

$$x' = p + \sqrt{p^2 - b^2}, \text{ and } x'' = p - \sqrt{p^2 - b^2}.$$

To construct these values of x. Construct the radical part of the values of x, as in the last case. Then, since p is positive, we lay off its value AD, from A towards the right. To AD, we add DB, which gives AB, for the first value of x. If from AD, we subtract DE, the remainder, AE, is the second value of x. Both values are positive, and are estimated in the same direction, from A to the right.

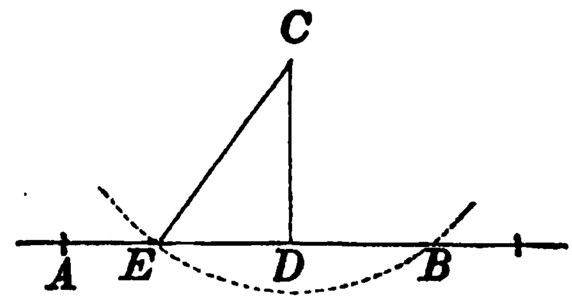


In the last two forms, if p and b are equal, the two values of x become equal to each other.*

The geometrical construction conforms to this result.

* Bourdon, Art. 116–117. University, Art. 146.

For, when $p = b$, the arc of the circle described with the centre C, will be tangent to AB, at D; and the two points E and B will unite, and each root will become equal to AD.



If b^2 is greater than p^2, the value of x, in the last two forms, will be imaginary.*

The geometrical construction also indicates this result. For, if b exceeds p, the circle described with the centre C, and radius equal to p, will not cut the line AB. Hence,

The imaginary roots of an equation give rise to conditions in the construction which cannot be fulfilled; and this should be so, since the imaginary roots can never appear, unless the conditions of the equation are inconsistent with each other.

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* Bourdon, Art. 116–117. University, Art. 146.

ANALYTICAL GEOMETRY.

BOOK I.

POINT AND STRAIGHT LINE—PROBLEMS—TRANSFORMATION OF CO-ORDINATES—POLAR CO-ORDINATES.

Definitions.

1. ANALYTICAL GEOMETRY is that branch of Mathematics which has for its object the determination of the forms and properties of the Geometrical Magnitudes, by means of Analysis.

2. In Analytical Geometry, the quantities considered may be divided into two classes:

1st. *Constant quantities, which preserve the same values in the same investigation;* and,

2d. *Variable quantities, which assume all possible values that will satisfy the equation which expresses the relation between them.*

The constants are denoted by the first letters of the alphabet, a, b, c, &c.; and the variables, by the final letters, x, y, z, &c.

3. The terms, *straight line*, and *plane*, are used in their most extensive signification.

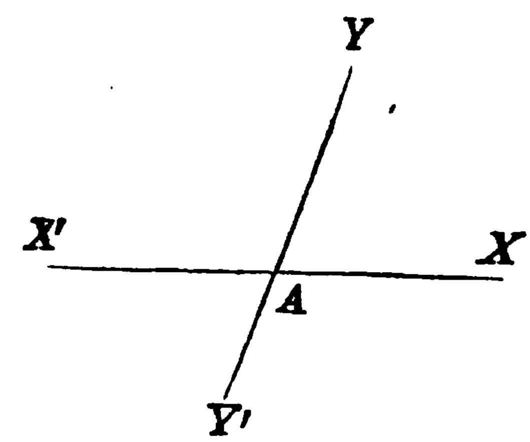
That is, the straight line is supposed to be indefinitely

prolonged, in both directions; and the plane, to be indefinitely extended.

Points in the same plane.

4. We shall first explain the manner of determining, by the algebraic symbols, the position of points in a given plane.

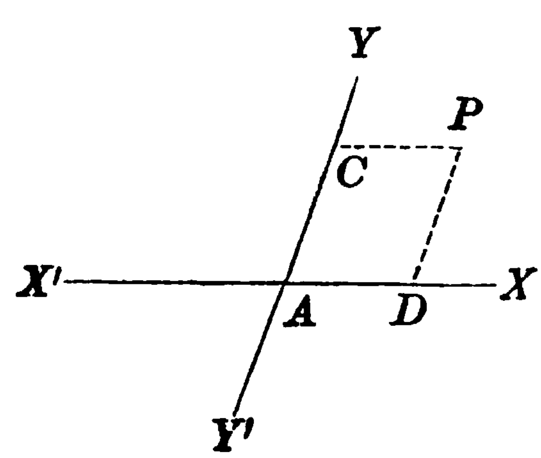
For this purpose, draw, in the plane, any two lines, as $X'AX$, YAY', intersecting at A, and making with each other a given angle, YAX.



The line $X'X$, is called the *axis of abscissas*, or the axis of X; and YY', the *axis of ordinates*, or the axis of Y. The two taken together, are called the *co-ordinate axes;* and the point A, where they intersect, is called the *origin of co-ordinates.* The angle YAX is called, the *first angle;* YAX', the second angle; $X'AY'$, the third angle; and $Y'AX$, the fourth angle.

First Angle.

5. Let P be any point in the given plane. Through P, draw PD, parallel to AY, and PC, parallel to AX. Then, AD, or CP, is called the *abscissa* of the point P; PD, or AC, is called the *ordinate* of P; and the lines PD, PC, taken together, are called the *co-ordinates* of the point P.



Hence, the abscissa of any point, is its distance from the axis of ordinates, measured on a line parallel to the axis of abscissas; and the ordinate of any point, is its distance from the axis of abscissas, measured on a line parallel to the axis of ordinates. The co-ordinates may also be measured on the axes themselves. For, AD, AC, are equal to the co-ordinates of the point P.

The co-ordinates of points are designated by the letters corresponding to the co-ordinate axes; that is, the abscissas are designated by the letter x, and the ordinates by the letter y.

1. If the co-ordinates of a point are known, the position of the point may be found. For, let us suppose that we know the co-ordinates of any point, as P. Then, from the origin A, lay off, on the axis of abscissas, a distance AD, equal to the known abscissa; and through D, draw a parallel to the axis of ordinates. Lay off, on the axis of ordinates, a distance AC, equal to the known ordinate, and through C, draw a parallel to the axis of abscissas; the point P, in which it meets DP, will be the position of the point.

2. When the co-ordinates of a point are known, the point is said to be *given*; and we have,

$$x = a, \text{ and } y = b;$$

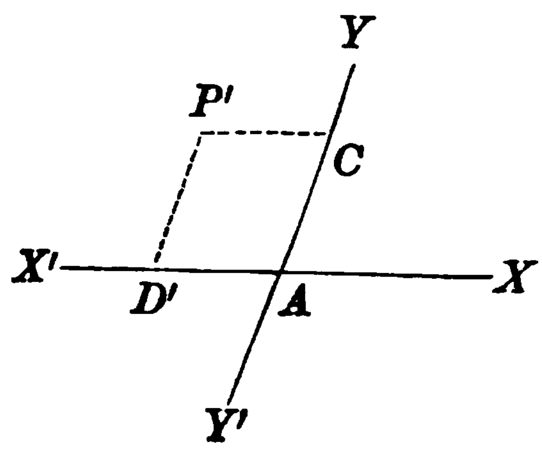
these are called, the *equations of the point.* Hence,

The equations of a point are the equations which express the distances of the point from the co-ordinate axes.

Second Angle.

6. Let us consider the given point P', in the second angle, YAX'.

The abscissa of this point is CP', or AD', and the ordinate, $P'D'$, or AC. Since distances estimated at the right of Y, have been regarded as positive, those at the left, are negative;* hence, the equations of the point P', are,

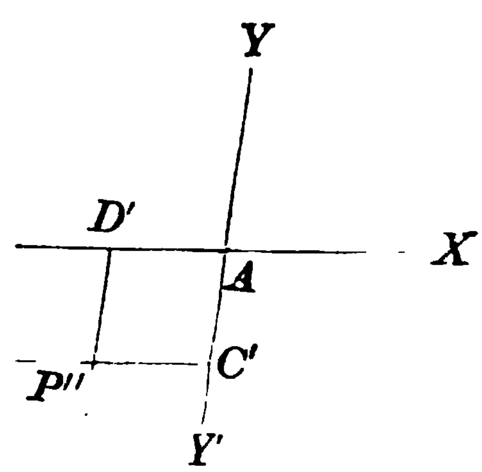


$$x = -a, \quad \text{and} \quad y = b.$$

Third Angle.

7. Let us consider the given point P'', in the third angle.

The abscissa of this point is $C'P''$, or AD', and negative. The ordinate is $D'P''$, or AC'. Since distances above the axis of X, have been regarded as positive, those below it are negative; hence, the equations of the point P'', are,



$$x = -a, \quad \text{and} \quad y = -b.$$

Fourth Angle.

8. Let us consider the given point P''', in the fourth angle.

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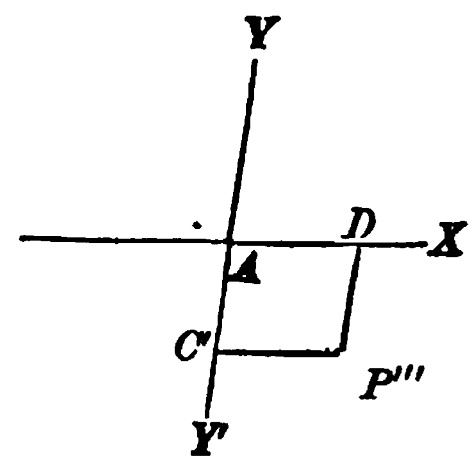
* Bourdon, Art. 89. University, Art. 96.



The abscissa of this point is $C'P'''$, or AD, and positive. The ordinate is DP''', or AC', and negative; hence, the equations of the point P''', are,

$$x = a, \text{ and } y = -b.$$

Therefore, the following are the equations of a point in each of the four angles:

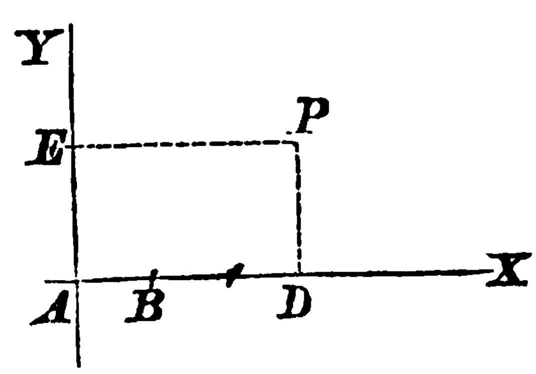
1st angle,	$x = +a,$	$y = +b.$
2d angle,	$x = -a,$	$y = +b.$
3d angle,	$x = -a,$	$y = -b.$
4th angle,	$x = +a,$	$y = -b.$

We see, by examining these results, that the signs of the abscissas in the different angles, correspond to the algebraic signs of the cosines, in the different quadrants of the circle; and that the signs of the ordinates, correspond to the algebraic signs of the sines.*

EXAMPLES IN CONSTRUCTION.

1. Determine the point whose equations are,

$$x = 3, \text{ and } y = 2.$$



Having drawn the co-ordinate axes, lay off, on the axis of X, a distance AB, to denote the unit of length. Then lay off, from A to D, three times the unit of length. From A, on the axis of

* Legendre, Trigonometry, Art. 16.

Y, lay off a distance AE, equal to twice the unit of length. Through D and E, draw parallels to the axes, and their point of intersection, P, will be the required point.

2. Determine the point whose co-ordinates are,

$$x = -5, \quad \text{and} \quad y = 4.$$

3. Determine the point whose co-ordinates are,

$$x = -7, \quad \text{and} \quad y = -8.$$

4. Determine the point whose co-ordinates are,

$$x = 4, \qquad \text{and} \quad y = -6.$$

Co-ordinate Axes, and Origin.

9. Since the ordinate of a point is its distance from the axis of X, the ordinate of any point of that axis must be zero. Hence, the equations of a *given point*, in the axis of X, will be,

$$x = \pm a, \quad \text{and} \quad y = 0;$$

the plus sign before a, being used when the point is at the right of the origin, and the minus sign, when it is at the left.

If we attribute to a, *all possible values* between 0 and $+\infty$, the equations will embrace all points of the axis of X, at the right of the origin; and if we give to a, all values between 0 and $-\infty$, they will embrace all points of the axis of X, at the left of the origin. Both these conditions are expressed by the simple phrase,

$$x \text{ indeterminate.}$$

Hence, for *all points* in the axis of X, we have,

$$x \text{ indeterminate,} \quad \text{and} \quad y = 0$$

10. For a *given point* of the axis of Y, we have,

$$x = 0, \quad \text{and} \quad y = \pm b;$$

the plus sign before b, being used when the point is above the axis of X, and the minus sign when it is below. For all points in the axis of Y, we have,

$$x = 0, \quad \text{and} \quad y \text{ indeterminate.}$$

11. Since the origin of co-ordinates is in the axis of Y, its abscissa is zero; and since it is in the axis of X, its ordinate is zero. Hence, the equations of the origin are,

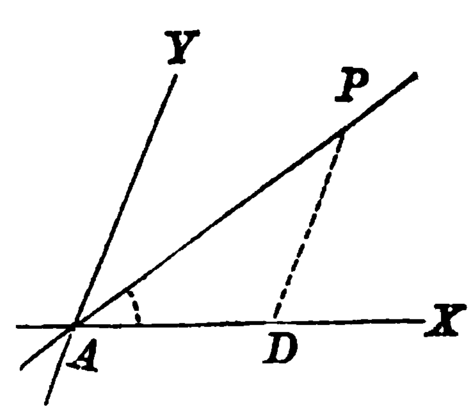
$$x = 0, \quad \text{and} \quad y = 0.$$

Straight lines in the same plane.

12. *The equation of a line, is an equation which expresses the relation between the co-ordinates of every point of the line.*

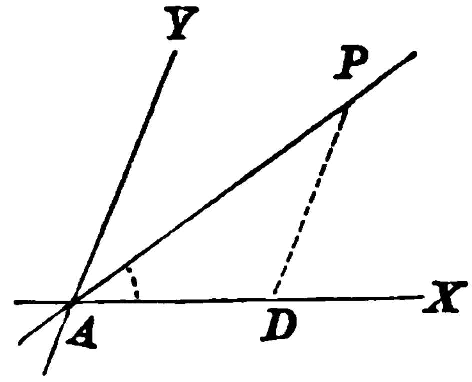
Equation of a straight line.

13. Let A be the origin of co-ordinates, and AX, AY, the co-ordinate axes. Through A, draw any straight line, as AP, making with the axis of X an angle denoted by α. Denote the angle YAX, included by the co-ordinate axes, by β.



Take any point of the line, as P, and draw PD parallel to the axis of Y: then, PD will be the ordinate, and AD the abscissa, of the point P.

Since PD is parallel to the axis of ordinates, the angle APD is equal to PAY: that is, equal to $\beta - \alpha$.



Since the sides of a triangle are to each other as the sines of their opposite angles,* we have

$$PD : AD :: \sin \alpha : \sin (\beta - \alpha).$$

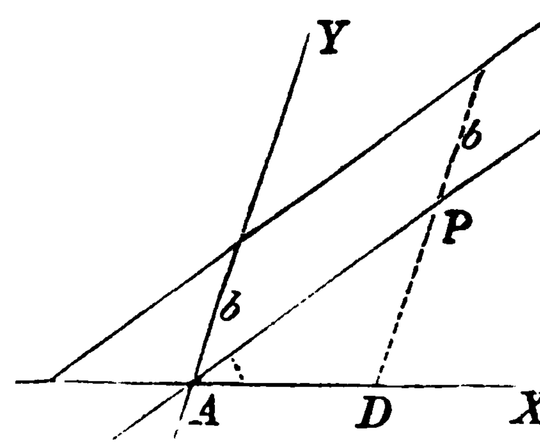
But PD is to AD, as any ordinate y of the line AP, to the corresponding abscissa x; therefore,

$$y : x :: \sin \alpha : \sin (\beta - \alpha),$$

which gives,

$$y = \frac{\sin \alpha}{\sin (\beta - \alpha)} x;$$

and this is the equation of the straight line AP, since it expresses the relation between the co-ordinates of every point of the line.



1. If we draw a line parallel to AP, cutting the axis of Y at a distance from the origin equal to b; it is plain, that for the same abscissa x, the ordinate y, of this new line, will exceed the ordinate y of the line through the origin, by the constant quantity b; hence, the equation of the parallel line will be,

$$y = \frac{\sin \alpha}{\sin (\beta - \alpha)} x + b.$$

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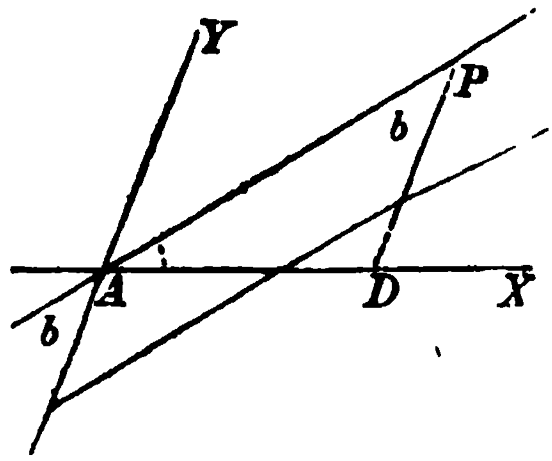
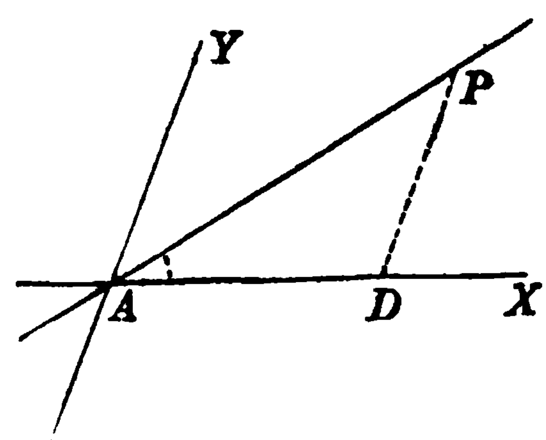
* Legendre, Trig. Art. 45.

2. If the parallel cuts the axis Y below the origin of co-ordinates, the value of y, in the second line, will be less than the value of y in the line AP, by the constant quantity b; and in that case, b becomes negative, and the general equation takes the form,



$$y = \frac{\sin \alpha}{\sin (\beta - \alpha)} x - b$$

3. Since the line PD is parallel to the axis of Y, the angle APD is equal to the angle PAY; hence, *the coefficient of* x *is the sine of the angle which the line makes with the axis of X, divided by the sine of the angle which it makes with the axis of Y.*



4. Thus far, we have supposed the co-ordinate axes to make an oblique angle with each other. It is, however, generally, most convenient to refer points and lines to co-ordinate axes which are at right angles.

If we suppose YAX to be a right angle,

$$\beta - \alpha = 90^\circ - \alpha,$$

and, $$\sin (\beta - \alpha) = \cos \alpha.*$$

* Legendre, Trig. Art. 8.

The equation of the straight line then takes the form,

$$y = \frac{\sin \alpha}{\cos \alpha} x \pm b;$$

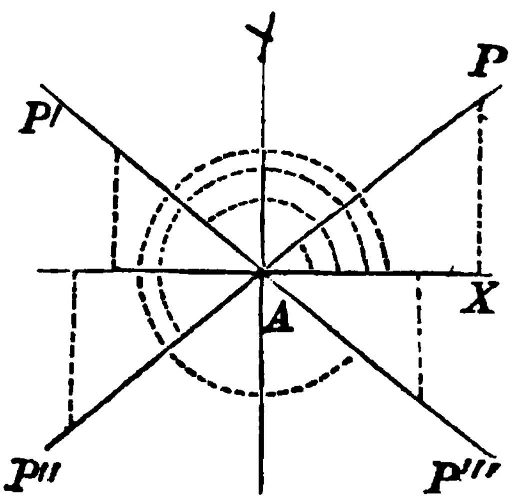
or,
$$y = \text{tang}\, \alpha x \pm b,$$

the tangent of α being calculated to the radius 1.

If we denote the tangent of α by a, the equation becomes,

$$y = ax \pm b.$$

Interpretation of the equation.



14. The line AP, passing through the origin of co-ordinates, and whose equation is $y = ax$, has been drawn in the first angle. But the equation is equally applicable to a line drawn in either of the other angles, if proper values and signs be attributed to the tangent, a. The angle, of which a is the tangent, is always estimated from the axis AX, around to the left.

1. If the line be drawn in the first angle, the tangent a is positive,* and the co-ordinates x and y, are both positive.

2. If the line be drawn in the second angle, the angle XAP' will fall in the second quadrant, and its tangent, a, will be negative.* But the abscissas of points in the second angle are also negative: hence, a and x are both negative: their product is, therefore, positive; hence, y is posi-

* Legendre, Trig. Art. 18.

tive, as it should be, since it represents the ordinates of points above the axis of abscissas.

3. If the line AP' be drawn in the third angle, the tangent a will be positive, since the angle falls in the third quadrant;* and since x is negative, the second member will be negative; hence, y will be negative, as it should be.

4. If the line AP'' be drawn in the fourth angle, the tangent a will be negative, since the angle falls in the fourth quadrant; and since x is positive, the second member will be negative, and therefore, y will be negative, as it should be.

As the same reasoning is applicable to the general form, the equation,

$$y = ax + b,$$

will represent every straight line which can be drawn on the plane of the co-ordinate axes, if proper values and signs are attributed to a and b.

5. The values of a and b are constant for the same straight line, but take different values when we pass from one line to another. They are called *arbitrary constants*, because values may be attributed to them at pleasure, when only the *form* of the equation is given.

6. If, in the equation of a straight line

$$y = ax + b,$$

any value be attributed to one of the variables, the other becomes determinate, and its value may be found from the equation.

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* Legendre, Trig. Art. 18

If, for example, we make,

$x = 1$,	we have,	$y = a + b$.
$x = 2$,	gives,	$y = 2a + b$.
$x = 3$,	gives,	$y = 3a + b$.
&c.,	&c.,	&c.

Or, we may attribute values to y, and find the corresponding values of x. If we make,

$y = 1$,	we have,	$x = \frac{1 - b}{a}$.
$y = 2$,	gives,	$x = \frac{2 - b}{a}$.
$y = 3$,	gives,	$x = \frac{3 - b}{a}$.

Construction of Straight Lines.

15. The *construction* of a line, represented by an equation, is the operation of drawing the line on the plane of the co-ordinate axes.

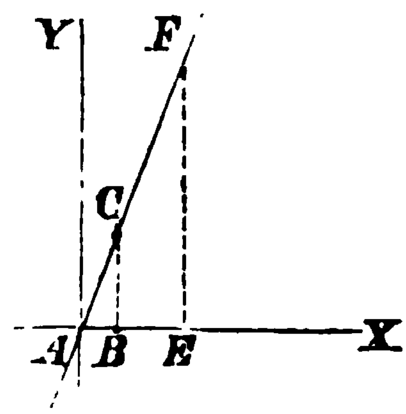
16. A line is said to be *given*, or *known*, when the form of its equation is given, and when the constants have *fixed* values. The position of the line is then determined, and the line can be constructed.

First Method.

1. Construct the line whose equation is,

$$y = 3x.$$

This line passes through the origin of co-ordinates, and makes with the axis of X, an angle whose tangent is 3. (Art. **13**.)

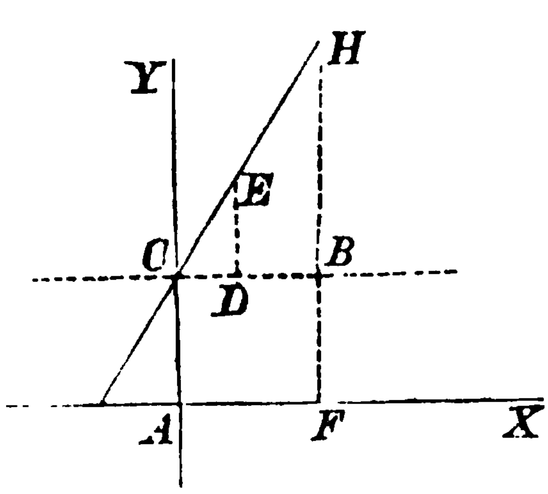


Having drawn the co-ordinate axes at right angles to each other, lay off from the origin, the unit of length, $AB = 1$. At B, draw BC perpendicular to the axis of X, and make it equal to 3: that is, to three times the unit of length; then, draw ACF.

Since the tangent of the angle at the base is equal to the perpendicular divided by the base,* BC will be the tangent of the angle BAC, to the radius 1; hence, ACF is the required line; for, any ordinate, as FE, will be equal to 3 times the abscissa, AE.

2. Construct the line whose equation is,

$$y = 2x + 4.$$



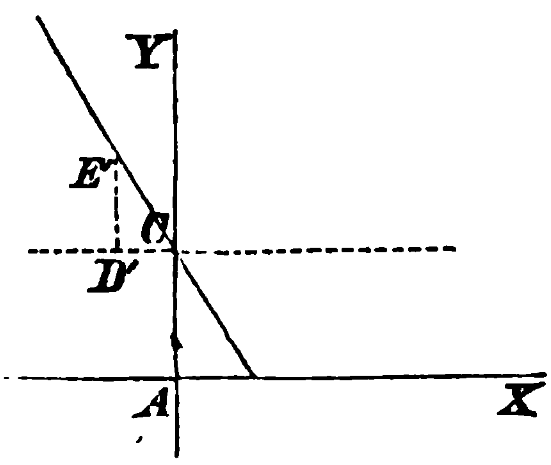
This line will cut the axis of Y, at a distance from the origin, above the axis of X, equal to 4 (Art. **13**). Having assumed the unit of length, lay it off 4 times, from A to C. Through C, draw a parallel to the axis of X. Then lay off CD, equal to the unit of length. Draw DE perpendicular to X, and make it equal to 2. The indefinite straight line passing through C and E, will be the line required. For, since DE is twice CD, HB will be twice CB or AF; hence, any ordinate, as HF, will be equal to twice the abscissa,

* Legendre, Trig. Art. 31.

AF, plus AC, which is 4. If the coefficient of x were negative, the equation would take the form,

$$y = -2x + 4;$$

the point D would then fall at D', and the line would take the direction CE'. Every straight line passing through the origin of co-ordinates will lie in the first and third angles when its coefficient is positive, and in the second and fourth, when it is negative.



If 4 were negative, the point C, would fall on the axis of Y, below the origin.

Second Method.

17. A point which is common to two straight lines will be at their intersection; and the co-ordinates of this point will satisfy the equations of both lines.

Conversely, if the equations of two straight lines be made simultaneous,* and combined, the results obtained will be the co-ordinates of the common point.

1. Construct the line whose equation is,

$$y = -6x + 12 \quad . \quad . \quad . \quad . \quad (1.)$$

If this equation be combined with the equation of the axis of X, which is,

$$x \text{ indeterminate, and } y = 0 \quad \text{(Art. 9)},$$

we shall have, $x = 2$,

* Bourdon, Art. 82. University, Art. 88.

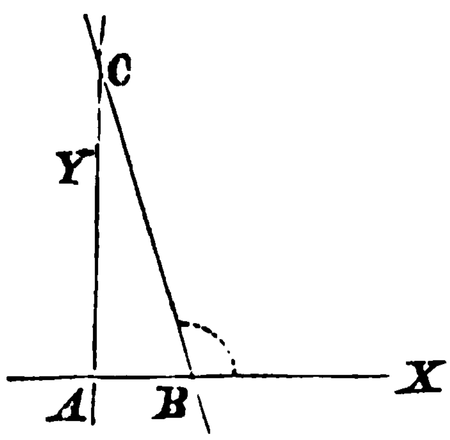
which is the abscissa of the point in which the line intersects the axis of X.

If we combine Equation (1), with the equation of the axis of Y, which is,

$$x = 0, \quad \text{and} \quad y \text{ indeterminate (Art. } \mathbf{10}\text{)},$$

we shall have, $$y = 12,$$

which is the ordinate of the point in which the line intersects the axis of Y.



Having drawn the co-ordinate axes, at right angles to each other, lay off, on the axis of X, AB equal to twice the unit of length; and on Y, AC equal to 12 times the unit of length, and then draw the line BC; this will be the line required, since two points determine the position of a straight line. The line makes an obtuse angle with the axis X, as it should do, since the coefficient of x is negative.

2. Construct the line whose equation is,

$$y = ax + b.$$

3. Construct the line whose equation is,

$$y = 2x + 5.$$

4. Construct the line whose equation is,

$$y = -x - 1.$$

5. Construct the line whose equation is,

$$y = -2x + 6.$$

Equation of the first degree between two variables.

18. The equation,

$$Ay + Bx + C = 0,$$

is the most general form of an equation of the first degree between two variables, since there is an absolute term C, and since each of the variables, y and x, has a coefficient.

This equation may be written under the form,

$$y = -\frac{B}{A}x - \frac{C}{A},$$

which becomes of the form already discussed, if we make,

$$-\frac{B}{A} = a, \text{ and } -\frac{C}{A} = b.$$

Having drawn the co-ordinate axes at right angles to each other, if we lay off on the axis of Y, a distance equal to $-\frac{C}{A}$, and through the point so determined, draw a line which shall make with the axis of X an angle whose tangent is $-\frac{B}{A}$; it will be the straight line whose equation is,

$$Ay + Bx + C = 0.$$

We may also put the equation under the form,

$$x = -\frac{A}{B}y - \frac{C}{B},$$

in which $-\frac{A}{B}$ is the tangent of the angle which the straight line makes with the axis of Y, and $-\frac{C}{B}$ the distance cut off from the axis of X, measured from the origin.

We may, therefore, state this general principle:

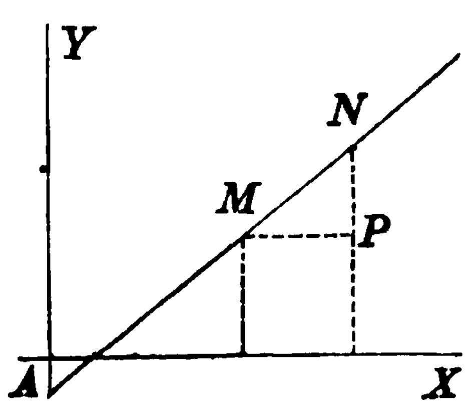
If an equation of the first degree between two variables, be solved with reference to either variable, the coefficient of the other variable will be the tangent of the angle which the line makes with the axis of that variable; and the absolute term will denote the distance cut off from the other axis.

Distance between two points.

19. A point is said to be given, when its co-ordinates are known. Known co-ordinates are usually designated by marking the letters, thus:

$$x', \ y'; \quad x'', \ y''; \quad x''', \ y''';$$

which are read, x prime, y prime, x second, y second, &c.



Let M and N be two given points. Designate the co-ordinates of M, by x', y', and the co-ordinates of N, by x'', y'', and MN, the distance between them, by D. Then,

$$MP = x'' - x', \quad \text{and} \quad NP = y'' - y'.$$

But, $$\overline{MN}^2 = \overline{MP}^2 + \overline{PN}^2;$$

hence, $$D^2 = (x'' - x')^2 + (y'' - y')^2,$$

or, $$D = \sqrt{(x'' - x')^2 + (y'' - y')^2}; \text{ that is,}$$

The distance between any two points is equal to the square root of the sum of the squares of the differences of their abscissas and ordinates.

1. If either of the points, as M, coincides with the origin, its equations will become,

$$x' = 0, \quad \text{and} \quad y' = 0,$$

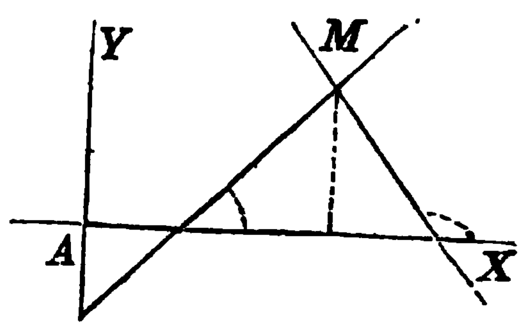
and we shall have,

$$D = \sqrt{x''^2 + y''^2},$$

a result which may be easily verified.

Equation of a line passing through a given point.

20. Let M be a given point, and designate its co-ordinates by x', y'. It is required to find the equation of a line which shall pass through this point.



The equation of every straight line is of the form,

$$y = ax + b. \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Since the required line is to pass through the point M, the co-ordinates of that point must satisfy Equation (1); hence, we have,

$$y' = ax' + b. \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

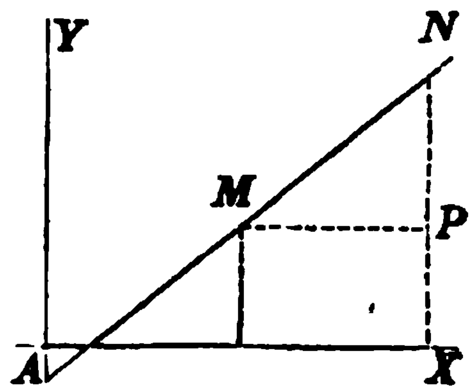
Combining Equations (1) and (2), we obtain,

$$y - y' = a(x - x'),$$

which is the equation of any line passing through the point whose co-ordinates are x' and y'.

1. In this equation, the tangent, a, is undetermined, as it should be, since an infinite number of lines may be drawn through the point M.

Equation of a line passing through two given points.



21. Let M and N, be two given points. Designate the co-ordinates of the first by x', y', and the co-ordinates of the second by x'', y''.

Since the required line is to pass through the point M, whose co-ordinates are x', y', its equation will be of the form (Art. 20),

$$y - y' = a(x - x') \quad . \quad . \quad . \quad . \quad (1.)$$

in which x and y are the co-ordinates of every point of the line.

Since the required line is also to pass through the point N, whose co-ordinates are x'', y'', these co-ordinates, when substituted for x and y, in Equation (1), will satisfy that equation; hence, we have,

$$y'' - y' = a(x'' - x');$$

from which we find, in known quantities,

$$a = \frac{y'' - y'}{x'' - x'}. \quad . \quad . \quad . \quad . \quad (2.)$$

Substituting this value in Equation (1), we have,

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'), \quad . \quad . \quad . \quad (3.)$$

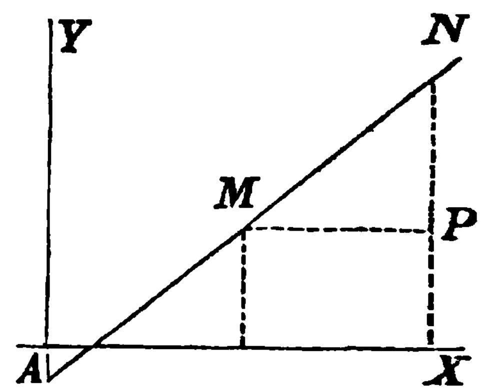
which is the equation of the required line.

Had we first passed the line through the point whose co-ordinates are x'', y'', Equation (1) would have taken the form,

$$y - y'' = a(x - x'');$$

and the equation of the required line would have been,

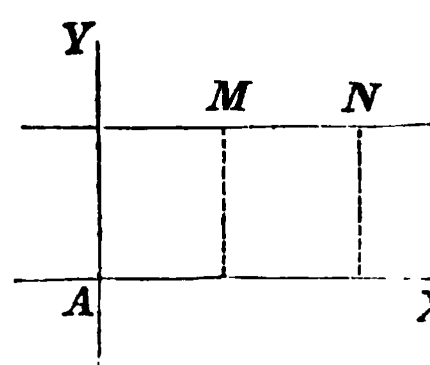
$$y - y'' = \frac{y'' - y'}{x'' - x'}(x - x'') \quad . \quad . \quad . \quad (4.)$$



1. The value of a, found in Equation (2), is easily verified. For, $y'' - y'$ is equal to NP, and $x'' - x'$ is equal to MP; hence,

$$\frac{NP}{MP} = \frac{y'' - y'}{x'' - x'},$$

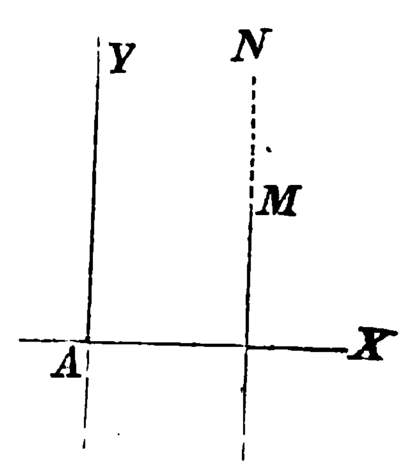
and, consequently, equal to the tangent of the angle NMP, to the radius 1.* Hence, a line passing through either of the points M or N, Equations (3) and (4), and making with the axis of X, an angle whose tangent is $NP \div MP$, will also pass through the other point.



2. If, in Equation (2), we suppose $y' = y''$, we shall have,

$$a = \frac{0}{x'' - x'} = 0;$$

and this is as it should be, since under this supposition, the line becomes parallel to the axis of X.



3. If we suppose $x' = x''$, in Equation (2), the ordinates y' and y'' being unequal, we shall have,

$$a = \frac{y'' - y'}{0};$$

therefore, a is infinite;† and hence, the line is perpendicular to the axis of X.‡

* Leg., Tr. Art. 31. † B., Art. 71. Un., Art. 72. ‡ Leg., Tr. Art. 24.

If we suppose $y' = y''$, and at the same time, $x' = x''$, the two points will coincide, and we shall have,

$$a = \frac{0}{0}.$$

Under these suppositions, a is indeterminate,* as it should be, since an infinite number of straight lines may be drawn through a single point.

Equation of a line parallel to a given line.

22. Let $$y = ax + b$$

be the equation of a given line (Art. **16**).

The equation of the required line will be of the form

$$y = a'x + b',$$

in which a' and b', are undetermined.

Two right lines are parallel, when they make the same angle with the axis of abscissas. Hence, if we make,

$$a' = a,$$

the second line will be parallel to the first; and its equation will be,

$$y = ax + b',$$

in which equation, b' is undetermined, as it should be, since an infinite number of lines may be drawn parallel to a given line.

1. If it be required that the parallel shall pass through a given point, its position will be entirely determined.

For, if the co-ordinates of the given point be denoted

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* Bourdon, Art. 71. University, Art. 72.

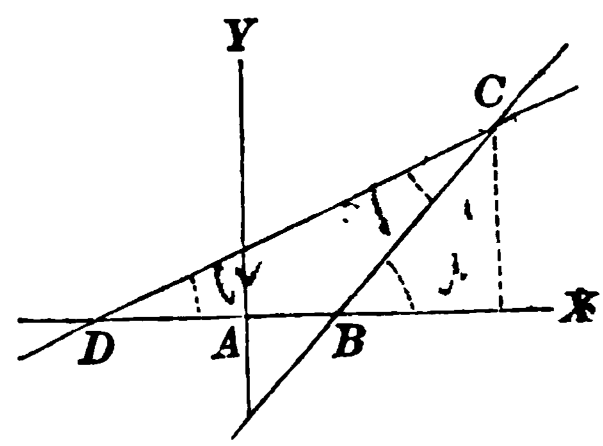
by x' and y', the equation of the parallel will take the form (Art. **21**),

$$y - y' = a(x - x'),$$

in which the quantities a, y', x', are known; hence, the position of the line is fixed.

Angle included between two given lines.

23. Let DC, BC, be the two given lines:



$$y = ax + b,$$

the equation of the 1st,

$$y = a'x + b',$$

the equation of the 2d, in which a, a', b, b', are known.

Denote the angles CDX and CBX, by α and α', and the angle DCB, by V.

Then, since $CBX = CDB + DCB$,*

we have, $$V = \alpha' - \alpha,$$

and, $$\text{tang } V = \text{tang } (\alpha' - \alpha) = \frac{\text{tang } \alpha' - \text{tang } \alpha}{1 + \text{tang } \alpha' \text{ tang } \alpha}$$

to the radius 1.†

Substituting for tang α', and tang α, their values a' and a, we have,

$$\text{tang } V = \frac{a' - a}{1 + a'a}.$$

* Legendre, Bk. I. Prop. 25. Cor. 6. † Trig. Art. 35.

1. If the lines become parallel, the angle V will be 0, and hence,

$$\text{tang } V = \frac{a' - a}{1 + a'a} = 0.*$$

Therefore, $a' - a = 0$; or, $a' = a$,

a relation already proved (Art. 22).

2. If the lines are perpendicular to each other, V will be equal to 90°, and its tangent infinite,† that is,

$$\text{tang } V = \frac{a' - a}{1 + a'a} = \infty;‡$$

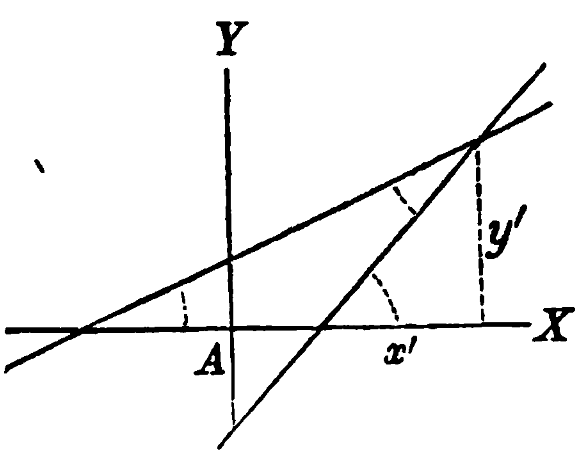
hence, $1 + a'a = 0$.

This last, *is the equation of condition, when two right lines are at right angles to each other.* If the tangent of one of the angles is known, the other can be found from the equation of condition.

Intersection of two lines.

24. Let $y = ax + b$, be the equation of the first;

and $y = a'x + b'$, the equation of the second.

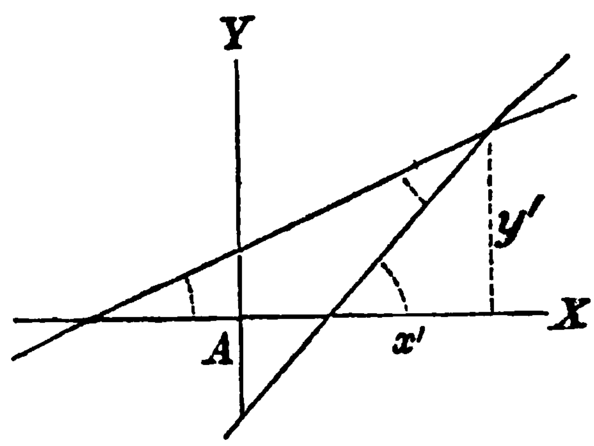


The point in which two straight lines intersect each other, is found, at the same time, on both of the lines; hence, its co-ordinates will satisfy both equations.

If, therefore, we suppose y

* Trig. Art. 24. † Leg., Trig. Art. 24. ‡ Bour., Art. 71. Univ. Art. 72.

and x, in the equation of the first line to become equal to y and x, in the equation of the second, the two equations will be simultaneous.



Combining the equations, under this supposition, and designating the co-ordinates of the point of intersection, by x' and y', we find,

$$x' = -\frac{(b - b')}{a - a'}, \quad \text{and} \quad y' = \frac{ab' - a'b}{a - a'}.$$

1. If, in the two last equations, we suppose $a = a'$, the values of x' and y' will both become infinite. The supposition of $a = a'$, is the condition when the two lines are parallel; and therefore, under this supposition, their point of intersection ought to be at an infinite distance from both the co-ordinate axes.

If, at the same time, we suppose $b = b'$, the values of x' and y', will become equal to 0 divided by 0, that is, indeterminate. The two suppositions will cause the lines to coincide; hence, their point of intersection ought to be indeterminate, since every point of either line will satisfy both equations.

A perpendicular from a given point to a given line.

25. Let $$y = ax + b, \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

be the equation of a given line, and x', y', the co-ordinates of a given point.

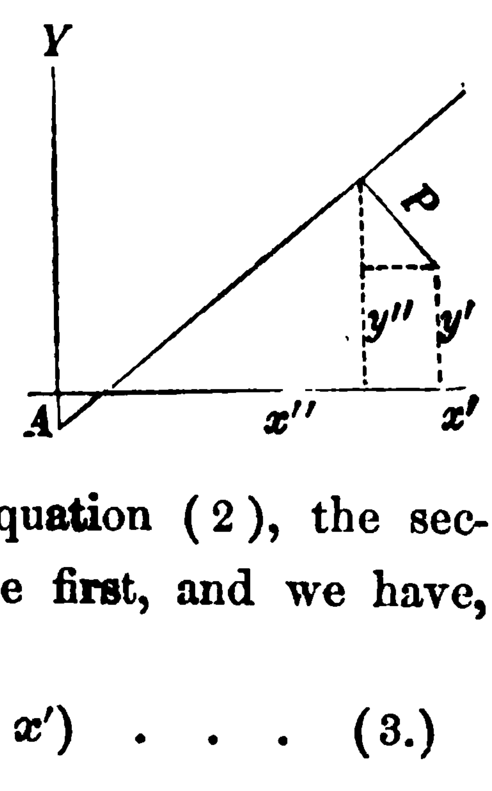
It is required to draw from this point, a line perpendicular to the given line, and to find the length of the perpendicular.

The equation of a line passing through a point, whose co-ordinates are x', y', (Art. **20**), is

$$y - y' = a'(x - x'), \quad . \quad . \quad . \quad . \quad (2.)$$

in which a' denotes the tangent of the angle which this second line makes with the axis of X.

But since this line is to be perpendicular to the given line, we have (Art. **23**),



$$1 + aa' = 0;$$

from which we have,

$$a' = -\frac{1}{a}.$$

Substituting this value for a', in Equation (2), the second line becomes perpendicular to the first, and we have,

$$y - y' = -\frac{1}{a}(x - x') \quad . \quad . \quad . \quad (3.)$$

It is now required to find the length of the perpendicular.

This is done, 1st, by finding the co-ordinates of the point in which the perpendicular intersects the given line; and 2d, by finding the differences between the co-ordinates of this point and the co-ordinates of the given point; 3d, by substituting these differences in Formula (Art. **19**).

Let us designate the co-ordinates of the point of intersection, by x'', y'', Then, since the point is on the given

line, its co-ordinates will satisfy the equation of that line, and we shall have,

$$y'' = ax'' + b; \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

and since the point is also on the perpendicular, its co-ordinates will also satisfy the equation of the perpendicular, and give,

$$y'' - y' = -\frac{1}{a}(x'' - x') \quad . \quad . \quad . \quad (5.)$$

If we eliminate x'', from these two equations, we shall have,

$$y'' = \frac{a^2y' + ax' + b}{1 + a^2}.$$

Subtracting y' from both members, we obtain,

$$y'' - y' = -\frac{y' - ax' - b}{1 + a^2} \quad . \quad . \quad . \quad . \quad (6.)$$

Substituting this value of $y'' - y'$, in Equation (5), we have,

$$x'' - x' = +\frac{a(y' - ax' - b)}{1 + a^2} \quad . \quad . \quad . \quad . \quad (7.)$$

Let us designate the length of the perpendicular, by P. Since the distance between two points, whose co-ordinates are, x'', y'', x', y', (Art. **19**), is

$$\sqrt{(x'' - x')^2 + (y'' - y')^2},$$

we have, by substituting for $x'' - x'$, and $y'' - y'$, their values found in Equations (7) and (6),

$$P = \frac{y' - ax' - b}{\sqrt{1 + a^2}}.$$

1. If the given point should fall on the given line,

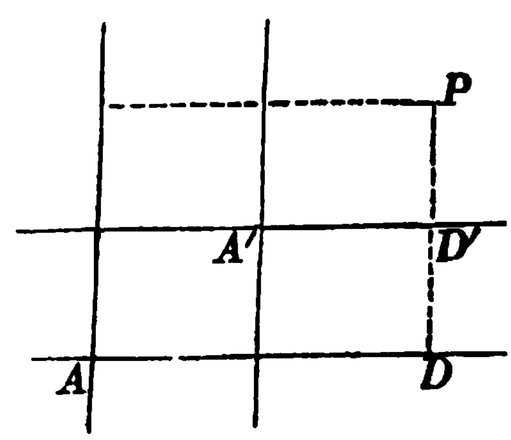
its co-ordinates would satisfy the equation of the line, and give,

$$y' = ax' + b.$$

This supposition would reduce the numerator of the value of P, to 0, and consequently, P would be equal to 0.

TRANSFORMATION OF CO-ORDINATES.

26. The equations of a point determine its position with respect to the co-ordinate axes (Art. **5**). The co-ordinate axes may be selected at pleasure, and any point may, at the same time, be referred to several sets of axes.



Let A, for example, be the origin of a system of co-ordinate axes, and A', any point whose co-ordinates are a and b.

Through A', draw two new axes, respectively parallel to the first.

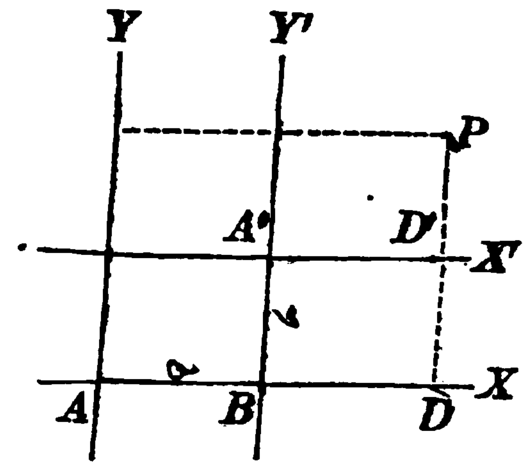
The co-ordinates of any point, as P, referred to the primitive system, are AD, PD; and its co-ordinates referred to the new system, are $A'D'$, PD'. The point P is equally determined, to which ever system it be referred.

27. It is often necessary, for reasons that will be hereafter explained, to change the reference of points from one system of co-ordinate axes to another. This is called, the *transformation of co-ordinates.* The axes to which the points are first referred, are called, *Primitive Axes;* and the second axes, to which they are referred, are called, *New Axes.*

In changing the reference of points from one system to another, all that is necessary, is to find the co-ordinates of the points referred to the primitive axes, in terms of the co-ordinates of the new origin and the co-ordinates of the points referred to the new axes.

To pass from a system of co-ordinate axes, to a parallel system.

28. Let A be the origin of the primitive system, and A' the origin of the new system. Suppose the co-ordinates of the origin A', to be, $AB = a$, and $BA' = b$; and let us designate the new axes, by X' and Y', and the co-ordinates of any point, referred to these axes, by x' and y'.



Then, assuming any point, as P, we shall have,

$$AD = AB + BD, \quad \text{and} \quad DP = DD' + D'P.$$

Now, since AD is the abscissa of P, and $BD = A'D'$, its abscissa referred to the new axes; and since PD is the ordinate of P, and $D'P$ its ordinate referred to the new axes, we have,

$$x = a + x', \quad \text{and} \quad y = b + y',$$

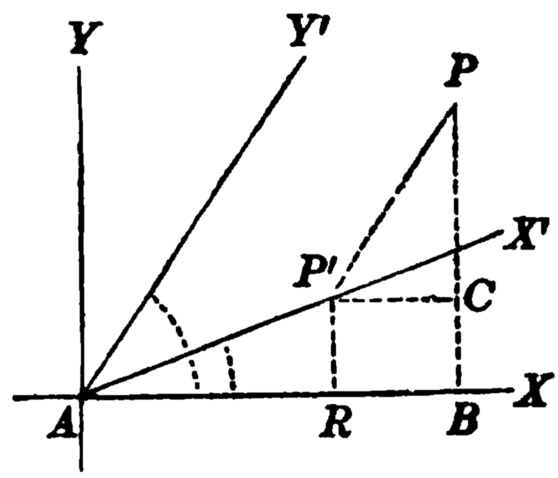
in which, the primitive co-ordinates of any point, are expressed in terms of the co-ordinates of the new origin, and the new co-ordinates of the same point.

1. The new origin may be placed in either of the four angles of the primitive axes, by attributing proper signs to its co-ordinates, a and b. It is also to be observed, that

x' and y' have the same algebraic signs, in the different angles of the new system, as have been attributed to x and y, in the corresponding angles of the primitive system.

To pass from a rectangular to an oblique system.

29. Let A be the common origin, AX, AY, the primitive axes, and AX', AY', the new axes; and let us designate, as before, the co-ordinates of points referred to the new axes, by x' and y'.



Denote the angle which the new axis of X' makes with the primitive axis of X, by α, and the angle which Y' makes with AX, by α'; and let P be any point in the plane of the axes. Through P, draw PB parallel to the axis of Y, and PP' parallel to the axis of Y'; draw also $P'R$ parallel to the axis of Y, and $P'C$ parallel to the axis of X.

Then, $$AB = AR + RB,$$

will be the abscissa of P, referred to the primitive axes;

and, $$PB = BC + CP,$$

will be its ordinate.

Also, AP' will be the abscissa of P, referred to the new system; then PP' will be its ordinate.

But, $$AR = AP' \cos \alpha,\text{*}$$

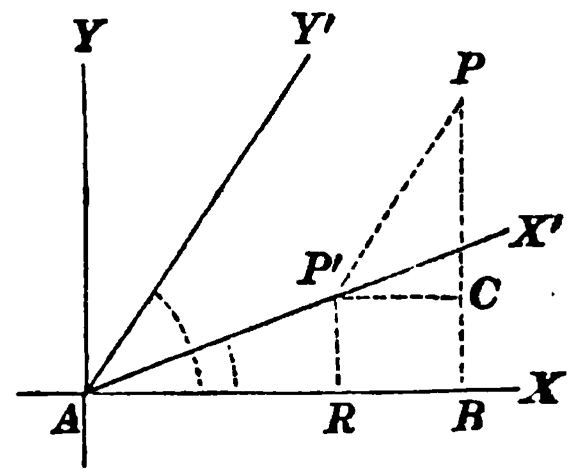
that is, $$AR = x' \cos \alpha,$$

and, $$RB = P'C = PP' \cos \alpha' = y' \cos \alpha';$$

* Trig. Art. 30.

hence, $$x = x' \cos\alpha + y' \cos\alpha'.$$

We also have,



$$P'R = CB = AP' \sin\alpha\,;$$

that is, $$CB = x' \sin\alpha, \text{ and,}$$

$$PC = PP' \sin\alpha' = y' \sin\alpha'\,;$$

hence, $$y = x' \sin\alpha + y' \sin\alpha'.$$

Hence, the formulas are,

$$x = x' \cos\alpha + y' \cos\alpha', \qquad y = x' \sin\alpha + y' \sin\alpha'.$$

1. If it were required, at the same time, to change the origin, to a point whose co-ordinates, referred to the primitive system, are a and b, the formulas would become,

$$x = a + x' \cos\alpha + y' \cos\alpha', \quad y = b + x' \sin\alpha + y' \sin\alpha'.$$

2. If $\alpha' - \alpha = 90°$, we have, $\sin\alpha' = \cos\alpha\,;$ $\cos\alpha' = -\sin\alpha\,;$* substituting these values in the last equation, we have,

$$x = a + x' \cos\alpha - y' \sin\alpha, \quad y = b + x' \sin\alpha + y' \cos\alpha,$$

which are the formulas for *passing from one system of rectangular co-ordinate axes to another.*

To pass from an oblique to a rectangular system.

30. The first set of formulas of the last Article, for a common origin, are,

$$x = x' \cos\alpha + y' \cos\alpha', \qquad y = x' \sin\alpha + y' \sin\alpha'.$$

* Trig. Art. 25.

If we regard the oblique as the primitive axes, it becomes necessary to find the co-ordinates of points referred to these axes, in terms of the rectangular co-ordinates, and the angles α and α'; that is, we must find the values of x' and y'.

If we multiply the first equation by the $\sin \alpha'$, and the second by $\cos \alpha'$, and then subtract them, and remember that, $\sin(\alpha' - \alpha) = \sin \alpha' \cos \alpha - \sin \alpha \cos \alpha'$, y' will be eliminated; and if x' be eliminated, in a similar manner, we shall obtain,

$$x' = \frac{x \sin \alpha' - y \cos \alpha'}{\sin(\alpha' - \alpha)}, \quad y' = \frac{y \cos \alpha - x \sin \alpha}{\sin(\alpha' - \alpha)}.$$

1. If the origin be changed, at the same time, to a point whose co-ordinates, with reference to the oblique system, are a and b, we shall have,

$$x' = a + \frac{x \sin \alpha' - y \cos \alpha'}{\sin(\alpha' - \alpha)}, \quad y' = b + \frac{y \cos \alpha - x \sin \alpha}{\sin(\alpha' - \alpha)}.$$

REMARKS.

31. The primitive co-ordinates of any point, determined with reference to a new system, depend for their values,

1st. On the position of the new origin:

2d. On the angles which the new axes make with the primitive axes: and,

3d. On the co-ordinates of the same point, referred to the new system.

32. The transformation of co-ordinates embraces two distinct classes of propositions:

1st. To transfer the reference of points from one system of co-ordinate axes to another system, which is known. In this case, the co-ordinates of the new origin, and the

angles which the new axes make with the primitive axes, are known.

2d. So to dispose of the new origin, and to give such directions to the new axes, as to cause the resulting equations to fulfil certain conditions, or to assume certain forms. In this case, the conditions imposed, determine the position of the new origin, and the directions of the new axes.

33. Since the primitive co-ordinates of points are always determined in linear functions of the new co-ordinates, that is, by equations of the first degree, the substitution of their values in the equation of any line, will not alter the degree of that equation; hence,

A given equation of a line, and its equation when referred to a new system of co-ordinate axes, will always be of the same degree.

34. We shall terminate this subject by a single example. Having given, the equation of a straight line,

$$y = a'x + b',$$

referred to rectangular co-ordinates, it is required to find its equation when the line is referred to oblique co-ordinates having a different origin. We have (Art. **29**—1),

$$x = a + x' \cos\alpha + y' \cos\alpha', \qquad y = b + x' \sin\alpha + y' \sin\alpha'.$$

Substituting these values for x and y, in the equation of the line, we have,

$$b + x' \sin\alpha + y' \sin\alpha' = a'(a + x' \cos\alpha + y' \cos\alpha') + b';$$

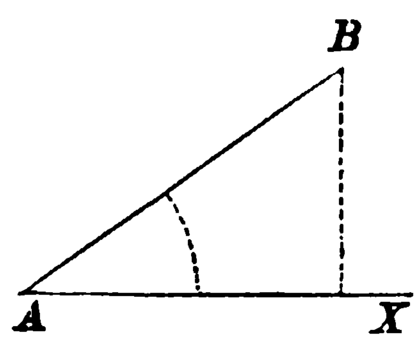
or, by reducing,

$$y' = \frac{a' \cos\alpha - \sin\alpha}{\sin\alpha' - a' \cos\alpha'}\, x' + \frac{aa' + b' - b}{\sin\alpha' - a' \cos\alpha'},$$

which is the equation of the straight line, referred to the oblique axes. The coefficient of x', is the sine of the angle which the line makes with the axis of X', divided by the sine of the angle which it makes with the axis of Y' (Art. **13**). The second term, in the second member, is the distance cut off from the axis of Y' (Art. **13**—3).

POLAR CO-ORDINATES.

35. We have seen, that the relative position of points and lines may be determined, analytically, by referring them to two co-ordinate axes. There are also other methods, by which they may likewise be determined.

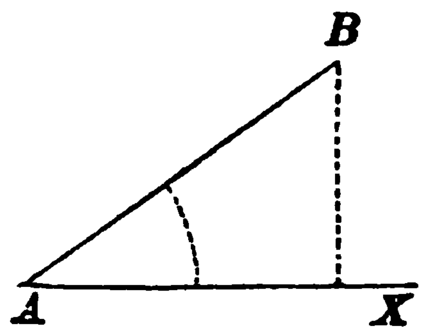


Assume, for example, any point, as A, and through it draw any straight line, as AX. If we suppose a straight line, as AB, to be turned around the point A, so as to make with AX all possible angles, from 0 to 360°, and suppose, at the same time, the line AB to increase or diminish at pleasure, the extremity B, may be made to occupy, in succession, every point of the plane.

Under this hypothesis, there are two variable quantities to be considered: 1st, the variable angle XAB; and 2d, the variable distance AB; and every point, in the plane, may be determined by attributing suitable values to these variables.

This method of determining points, by a variable angle and a variable distance, is called the *system of polar co-ordinates*. The variable distance AB, is called the *radius-vector;* and the fixed point A, from which it is estimated, is called the *pole*.

Designate the variable angle XAB, by v, the radius-vector AB, by r, and the co-ordinates of the point B, referred to rectangular axes, by x and y; then, if the origin of the rectangular axes be at A, we shall have,



$$x = r \cos v,^*$$

and,
$$y = r \sin v.^*$$

From the first equation, we have,

$$r = \frac{x}{\cos v}.$$

Now, since x, and the $\cos v$, are both positive in the first and fourth angles, and both negative in the second and third, they will always be affected with the same sign; and hence, the sign of r will be constantly positive; consequently,

A negative value of the radius-vector can never enter into the analysis.

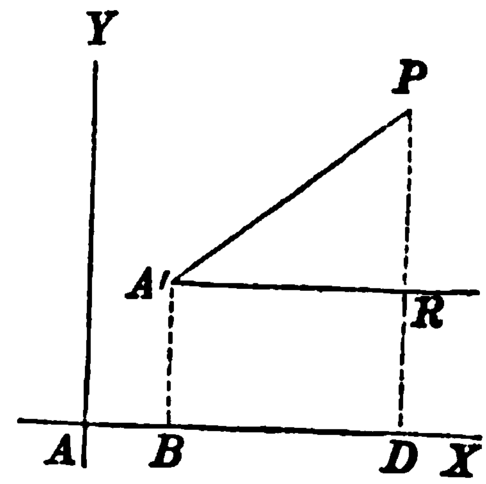
If, therefore, such a value should be obtained, we infer, that incompatible conditions have been introduced into the equations; and hence, *all negative values of the radius-vector must be rejected.*

To pass from a rectangular to a polar system.

36. Let A be the origin of the co-ordinate axes, A' the pole, $A'R$, parallel to AX, the line from which the varia-

* Legendre, Trig. 30.

ble angles are estimated, and $A'P$, the radius-vector of the point P. Let the co-ordinates of the pole A', be denoted by a and b.



Now, $A'R = r \cos v$,

and, $PR = r \sin v$.

But, $AD = AB + BD$,

and, $PD = DR + PR$;

hence, $x = a + r \cos v$,

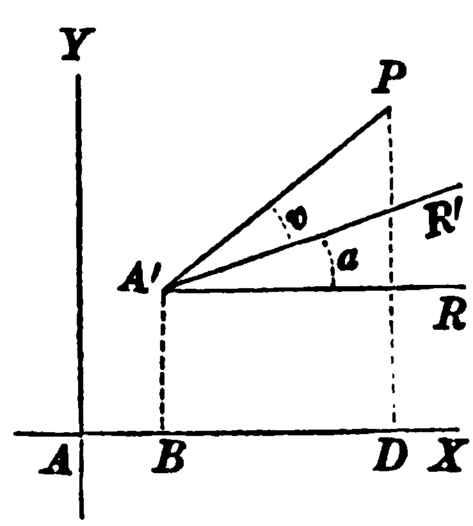
and, $y = b + r \sin v$;

which are the required formulas.

1. If the pole A', be placed at the origin A, the equations will become,

$$x = r \cos v, \quad y = r \sin v.$$

2. If, instead of estimating the variable angle v, from the line $A'R$, parallel to AX, it be estimated from $A'R'$, which makes with AX a given angle, $\pm\,\alpha$, the equations will become,



$$x = a + r \cos (v \pm \alpha),$$

$$y = b + r \sin (v \pm \alpha).$$

BOOK II.

OF THE CIRCLE.

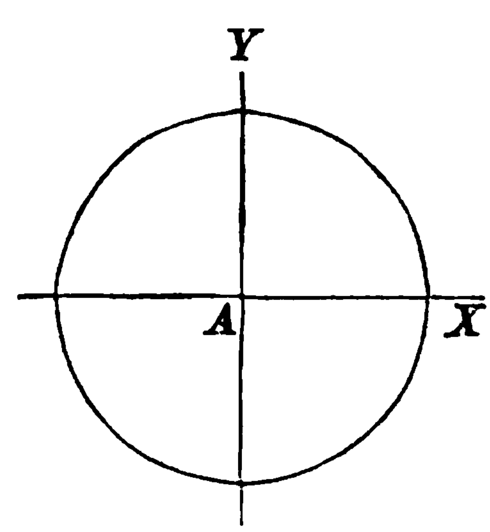
1. THE equation of a line expresses the relation which exists between the co-ordinates of every point of the line (Art. 12).

2. Lines are divided into different orders, according to the degree of their equations. For example, the right line is a line of the first order, since its equation is of the first degree. The circumference of the circle is a line of the second order, its equation being of the second degree; and if the equation of a line were of the third degree, the line would be of the third order.

3. The *Interpretation of an equation*, consists in classing the line which the equation represents; in determining its position, its form, its limits, and the points in which it intersects the co-ordinate axes.

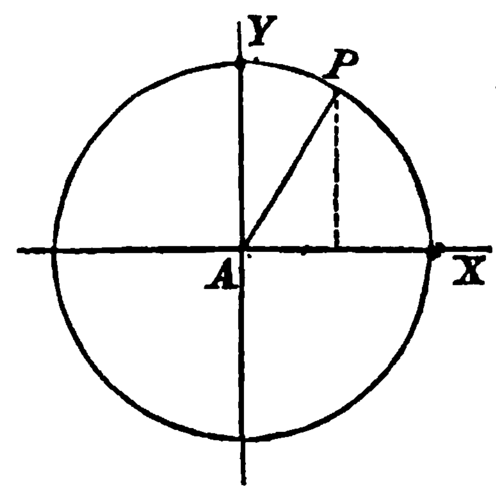
Equation of the circumference of a circle.

4. Let A be the origin of co-ordinates, and AX, AY, the co-ordinate axes.



It is required to find the equation of a curve such, that all its points shall be at a given distance from the origin A. Let R

denote the distance, and x and y, the co-ordinates of any point of the curve, as P. The square of the distance from the origin to any point, whose co-ordinates are x and y, is,



$$x^2 + y^2;$$

hence, $$x^2 + y^2 = R^2,$$

which is the equation required.

Interpretation.

1. To interpret the equation, we begin by determining the points in which the circumference cuts the co-ordinate axes.

The co-ordinates of these points must satisfy, at the same time, both the equation of the circle, and the equations of the axes.

The equations of the axis of X (Bk. I., Art. 9), being

$$x \text{ indeterminate, and } y = 0;$$

if we make $y = 0$, in the equation of the circle, the corresponding values of x will be the abscissas of those points which are common to the circumference and the axis of X; that is,

$$x = \pm R;$$

which shows that the curve cuts the axis of abscissas in two points, one on each side of the origin, and each at a distance from it equal to the radius of the circle.

2. To find the points in which the circumference cuts the axis Y, make $x = 0$, and there results,

$$y = \pm R;$$

the axis of Y, therefore, intersects the circumference in two points, equally distant from the origin, one above the axis of X, and the other below it.

3. To trace the curve between these points, find the value of y from the equation of the circle, which gives,

$$y = \pm\sqrt{R^2 - x^2}.$$

Now, since every value for x, gives for y two equal values, with contrary signs, it follows that the curve is symmetrical with respect to the axis of X; and in the same manner, it may be shown to be symmetrical with respect to the axis of Y.

Beginning at the point where $x = 0$, we have,

$$y = \pm R.$$

The values of y then decrease, numerically, as x increases numerically; and when x becomes equal to $\pm R$, we have,

$$y = 0;$$

hence, the curve intersects the co-ordinate axes in four points, at a distance from the origin, equal to R.

4. If x becomes greater than $\pm R$, the values of y become imaginary, which shows that the curve is limited both in the direction of x positive, and of x negative.

By placing the equation under the form,

$$x = \pm\sqrt{R^2 - y^2},$$

we may show, that the circumference is also limited in the direction of y positive, and in that of y negative.

5. By attributing a particular value to either of the variables, in the equation,

$$y = \pm\sqrt{R^2 - x^2},$$

the corresponding values of the other variable may be found.

If we suppose $R = 1$, and then make,

$x = 0,$	we have,	$y = \pm 1.$
$x = \frac{1}{2},$	gives,	$y = \pm\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}.$
$x = \frac{3}{4},$	gives,	$y = \pm\sqrt{\frac{7}{16}} = \frac{1}{4}\sqrt{7}.$
&c.,	&c.,	&c.

6. If, in the equation,

$$x^2 + y^2 = R^2,$$

x and y denote the co-ordinates of a point within the circumference, the equality will be destroyed, and $x^2 + y^2$, will be less than R^2, and we shall have,

$$x^2 + y^2 - R^2 < 0;$$

that is, negative.

For a point on the curve,

$$x^2 + y^2 - R^2 = 0;$$

and for a point without the curve,

$$x^2 + y^2 - R^2 > 0;$$

that is, positive.

7. The equation,

$$y^2 = R^2 - x^2,$$

may be put under the form,

$$y^2 = (R + x)(R - x),$$

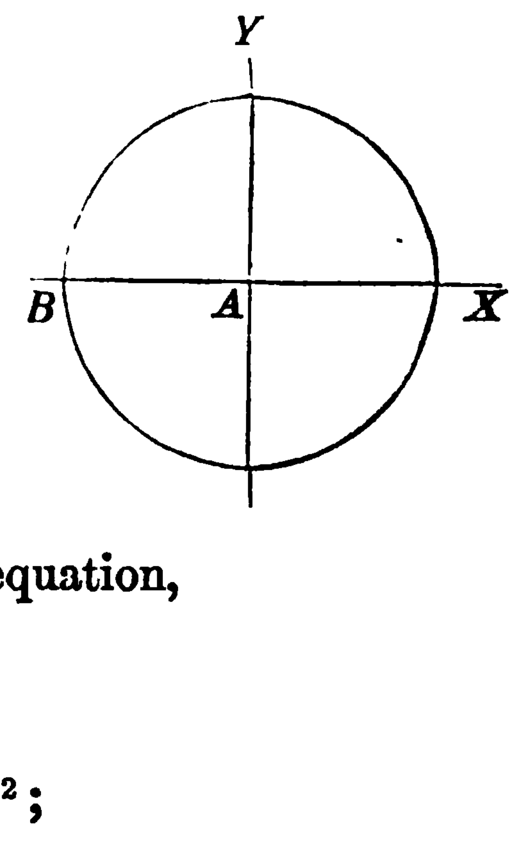
in which the factors, $R + x$, and $R - x$, are the two segments into which the ordinate y divides the diameter; this ordinate is, therefore, a mean proportional between the two segments.*

8. The equation of the circle may also be placed under another form, by transferring the origin of co-ordinates, from the centre to a point of the circumference.

For this transformation, we have the Formulas (Bk. I., Art. 28),

$$x = a + x', \quad \text{and} \quad y = b + y'.$$

Let the origin be transferred to B.



The co-ordinates of this point are,

$$a = -R, \quad \text{and} \quad b = 0;$$

hence,

$$x = -R + x', \quad \text{and} \quad y = y'.$$

Substituting these values in the equation,

$$y^2 = R^2 - x^2,$$

we obtain,

$$y'^2 = 2Rx' - x'^2;$$

or, omitting the accents,

$$y^2 = 2Rx - x^2;$$

which is the equation of the circle when the origin of co-ordinates is in the circumference.

9. When the absolute term in the equation of a line is wanting, the line will pass through the origin of co-ordinates.

* Legendre, Bk. IV. Prop. 23. Cor.

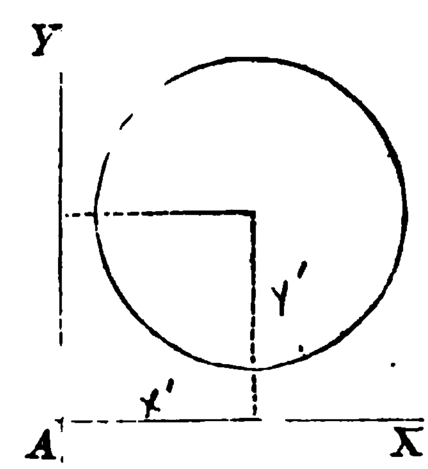
For, the co-ordinates of the origin are,

$$x = 0, \text{ and } y = 0;$$

these values being substituted in any equation wanting the absolute term, will reduce both members to 0; hence, the co-ordinates of the origin will satisfy the equation of the line; and, therefore, the line will pass through the origin.

5. There is yet a more general form under which the equation of a circle may be expressed.

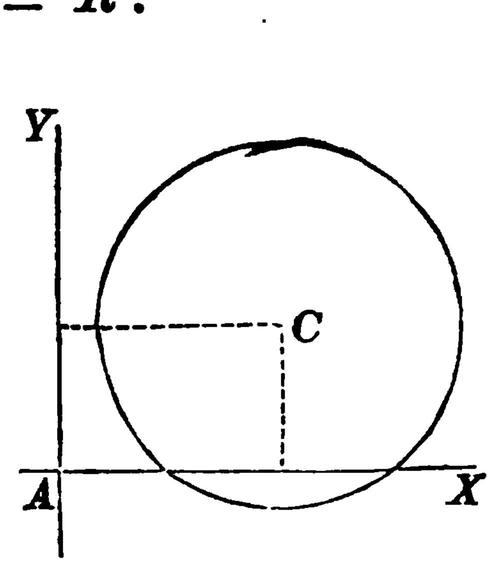
The characteristic property of the circumference of a circle is, that all its points are at an equal distance from the centre. To express this property, *analytically*, and in a general manner, designate the co-ordinates of the centre by x' and y'; the co-ordinates of any point of the circumference, by x and y, and the radius by R.



The distance from any point, whose co-ordinates are x', y', to a point whose co-ordinates are x and y (Bk. I., Art. **19**), is,

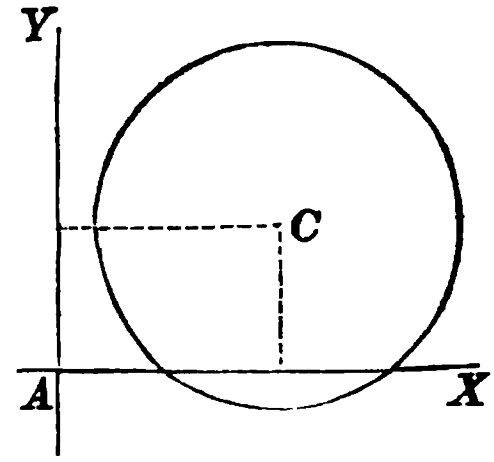
$$(x - x')^2 + (y - y')^2 = R^2.$$

This, therefore, is the most general equation of the circle referred to rectangular co-ordinates. By attributing proper values and signs to x' and y', the centre may be placed at any point in the plane of the co-ordinate axes.



1. To find the points in which

the circumference intersects the axis of X, make $y = 0$, and we have,



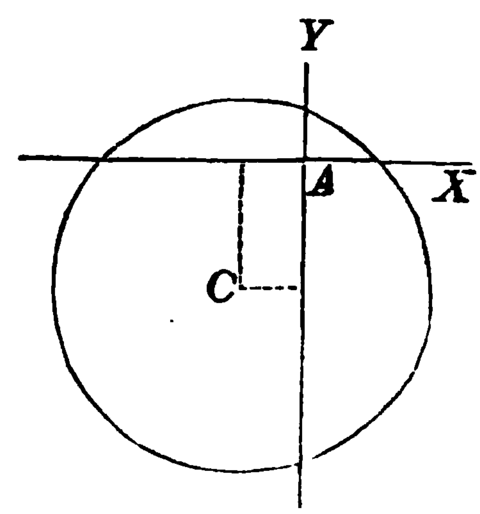
$$x = x' \pm \sqrt{R^2 - y'^2},$$

from which we see, that the values of x will become imaginary when y' exceeds R, and it is plain that in that case there will be no intersection.

2. To find the points in which the circumference intersects the axis of Y, make $x = 0$, and we have,

$$y = y' \pm \sqrt{R^2 - x'^2},$$

in which the values of y will be imaginary, if x' exceeds R.



3. If the co-ordinates of the centre of a circle are,

$$x' = -2, \text{ and } y' = -4,$$

and the radius equal to 6, its equation will be,

$$(x + 2)^2 + (y + 4)^2 = 36,$$

from which the circumference may be readily described.

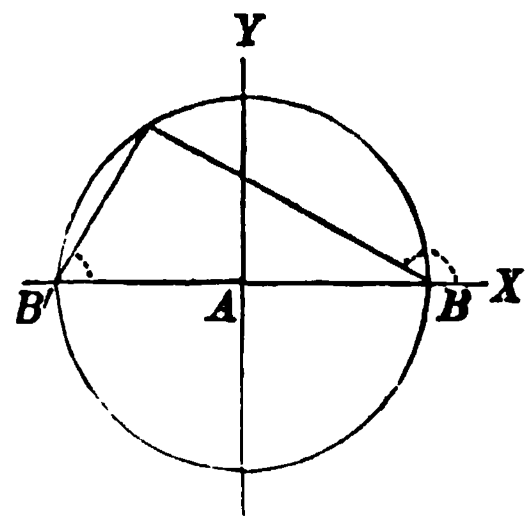
Find the points in which it cuts the co-ordinate axes.

Supplementary chords.

6. Supplementary Chords, are pairs of chords drawn through the extremities of a diameter, and intersecting each other on the curve.

Supplementary chords of the circle are at right angles.

7. Let A be the origin of co-ordinates, and B and B', the extremities of a diameter.



The equation of a straight line passing through a given point, is of the form (Bk. I., Art. **20**),

$$y - y' = a(x - x').$$

If the line passes through the point B, whose co-ordinates are $x' = +R$, and $y' = 0$, its equation will be,

$$y = a(x - R) \quad . \; . \; . \; . \; . \quad (1.)$$

For a like reason, the equation of a straight line passing through B', whose co-ordinates are $x' = -R$, and $y' = 0$, is,

$$y = a'(x + R) \quad . \; . \; . \; . \; . \quad (2.)$$

If these two lines intersect each other, the co-ordinates of their point of intersection will satisfy both equations. Hence, if we suppose x, in one equation, to be equal to x in the other, and y equal to y, and then combine the equations by multiplying them together, member by member, the resulting equation,

$$y^2 = aa'(x^2 - R^2), \quad . \; . \; . \; . \; . \quad (3.)$$

will express the condition, that the two straight lines shall intersect on the plane of the co-ordinate axes.

But, if the point of intersection is in the circumference of the circle, x and y must satisfy the equation,

$$x^2 + y^2 = R^2, \quad . \; . \; . \; . \; . \quad (4.)$$

or, $$y^2 = R^2 - x^2 = -1(x^2 - R^2).$$

Hence, $$aa' = -1, \quad \text{or,} \quad aa' + 1 = 0.$$

The two supplementary chords, therefore, are at right angles (Bk. I., Art. **23**—2).

1. In the equation of condition,

$$aa' + 1 = 0,$$

the two tangents a and a', are undetermined; there are, therefore, an infinite number of values which may be attributed to either of them, that will satisfy the equation; which shows, that there is an indefinite number of supplementary chords that may be drawn through the extremities of the same diameter, each pair of which will be at right angles.

2. If it be required, that one of the supplementary chords shall make a given angle with the axis of X, its tangent a or a', becomes known; and then, the value of the other tangent may be found from the equation of condition.

3. If either a or a' is equal to 0, the other will be infinite; which shows, that if one of the chords coincides with the axis of X, the other will become perpendicular to it.

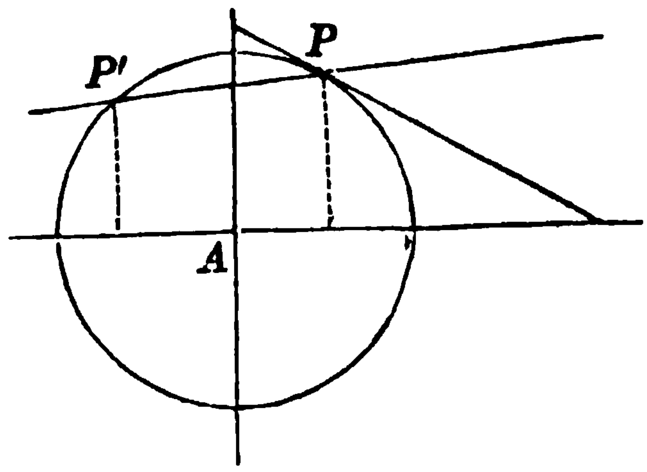
Tangent line to the circle.

8. It is required to find the equation of a tangent line to a circle.

Let A be the origin of co-ordinates, and,

$$x^2 + y^2 = R^2, \quad \ldots\ldots \quad (1.)$$

the equation of the circle.



Take any point of the circumference, as P, and designate its co-ordinates by x'', y''. Through this point draw a secant line; its equation will be of the form (Bk. I., Art. **20**),

$$y - y'' = a(x - x'') \quad . \quad . \quad . \quad . \quad (2.)$$

It is required to find the value of a, when the secant line PP' becomes tangent to the circumference.

Since the point P is in the circumference, its co-ordinates will satisfy the equation of the circle, and we shall have,

$$x''^2 + y''^2 = R^2 \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Subtracting Equation (3) from (1), member from member, we obtain,

$$x^2 - x''^2 + y^2 - y''^2 = 0,$$

or, $$(x + x'')(x - x'') + (y + y'')(y - y'') = 0 \quad . \quad (4.)$$

in which equation, x and y are the co-ordinates of any point of the circumference.

If Equation (4) be combined with (2), x and y, in the resulting equation, will be the co-ordinates of the points in which the secant intersects the circumference. The equations are most readily combined, by substituting for $y - y''$, in Equation (4), the value of $y - y''$ in Equation (2). Making the substitution, we obtain,

$$(x + x'')(x - x'') + (y + y'')\,a(x - x'') = 0,$$

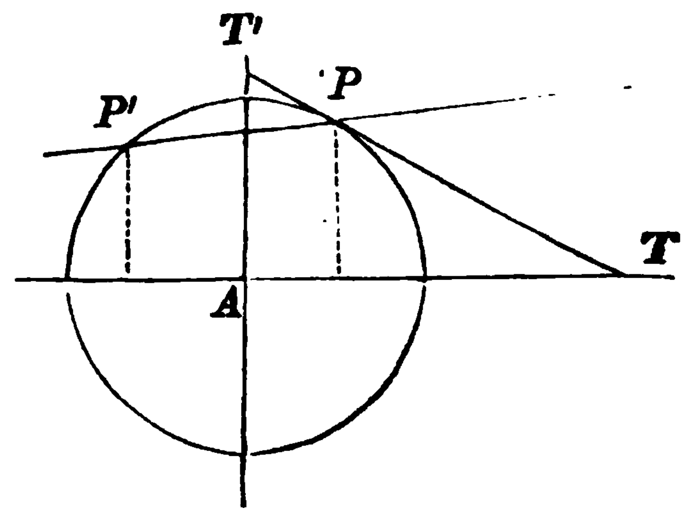
and, by factoring, we have,

$$(x - x'') \times [x + x'' + a(y + y'')] = 0,$$

which is satisfied by making,

$$x - x'' = 0, \quad \text{or,} \quad x + x'' + a(y + y'') = 0.$$

In the first equation, x denotes the abscissa of the point P; in the second, of P'.



If we suppose the secant PP' to turn around the point P, the point P' will approach P; and when P' shall coincide with P, the secant line will become tangent to the circumference. When this takes place, we shall have,

$$x = x'', \quad \text{and} \quad y = y'',$$

and the second equation will give,

$$a = -\frac{x''}{y''},$$

x'', y'', being the co-ordinates of the point of contact.

Substituting this value in Equation (2), we have,

$$y - y'' = -\frac{x''}{y''}(x - x'');$$

or, by reducing,

$$yy'' - y''^2 = -x''x + x''^2,$$

or,
$$yy'' + xx'' = y''^2 + x''^2,$$

or,
$$yy'' + xx'' = R^2;$$

in which x and y are the general co-ordinates of the tangent line.

1. For the point in which the tangent intersects the axis of Y, we have $x = 0$, and,

$$y = \frac{R^2}{y''} = AT'.$$

2. For the point in which the tangent intersects the axis of X, we have, $y = 0$, and,

$$x = \frac{R^2}{x''} = AT.$$

Normal line.

9. A NORMAL LINE to a curve, is a line perpendicular to the tangent at the point of contact.

Every normal line, in a circle, passes through the centre.

10. The tangent of the angle which the tangent line to a circle makes with the axis of X (Art. **8**), is,

$$a = -\frac{x''}{y''} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

The equation of any straight line passing through the point of tangency will be of the form,

$$y - y'' = a'(x - x'').$$

The equation of condition requiring this line to be perpendicular to the tangent (Bk. I., Art. **23**—2), is,

$$aa' + 1 = 0, \quad \text{or,} \quad a' = -\frac{1}{a} \quad . \quad . \quad (2.)$$

Substituting for a its value in Equation (1), we have,

$$a' = \frac{y''}{x''}.$$

The equation of the normal, therefore, becomes,

$$y - y'' = \frac{y''}{x''}(x - x''),$$

or, by reducing,

$$yx'' - y''x = 0; \quad \text{or,} \quad y = \frac{y''}{x''}x;$$

and since this equation has no absolute term (Art. 4—9), the line passes through the origin of co-ordinates. We have thus proved a property well known in Elementary Geometry, viz.: that a line perpendicular to the tangent at the point of contact, passes through the centre of the circle.

Polar equation.

11. The polar equation of a curve, is the equation which is obtained by referring the curve to a fixed point and a given straight line. The fixed point is called the *pole*; the variable distance, from the pole to any point of the curve, is called the *radius-vector*; and the angle which the radius-vector makes with the given straight line, is called the *variable angle*.

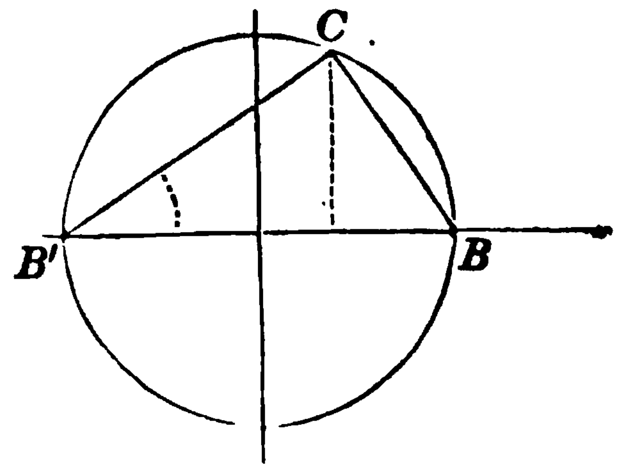
Polar equation of the circle.

12. Let it be required to find the polar equation of the circle, when the pole is in the circumference.

The equation of the circle, referred to rectangular co-ordinates, when the origin is in the circumference, as at B', (Art. 4—8), is,

$$y^2 = 2Rx - x^2 \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

If B' is the pole of the polar co-ordinates, we have (Bk. I., Art. **36**—1),



$x = r\cos v$, and $y = r\sin v$.

Substituting these values of x and y, in Equation (1), we have,

$$r^2\sin^2 v = 2Rr\cos v - r^2\cos^2 v.$$

Transposing, and remembering that,

$$\sin^2 v + \cos^2 v = 1,$$

we have, $$r^2 - 2Rr\cos v = 0;$$

which is the polar equation of the circle when the pole is at B', and the angle v, estimated from the axis of X.

Interpretation of the equation.

13. Since the polar equation,

$$r^2 - 2Rr\cos v = 0,$$

has no absolute term, one of the roots is equal to 0;* which ought to be the case, since the pole is on the curve (Art. **4**—9).

Dividing by this value of r, we obtain for the other value,

$$r = 2R\cos v.$$

This value of r will be positive, when the $\cos v$ is positive; and negative, when the $\cos v$ is negative. But the

* Bourdon, Art. 251. University, Art. 193.

negative values of the radius-vector must be rejected, since they cannot enter into the analysis (Bk. I., Art. 35).

The figure also indicates the same result. For, the $\cos v$ is positive in the first and fourth quadrants; hence, the radius-vector is positive when it falls in the first or fourth angle. The $\cos v$, is negative in the second and third quadrants; hence, the radius-vector is negative when it falls in the second or third angle.

For $v = 0$, the $\cos v = 1$, and we have,

$$r = 2R = B'B.$$

When v increases from 0 to 90°, the radius-vector continues positive, and determines all the points in the semi-circumference BCB'.

This may also be verified. For, in the right-angled triangle $B'CB$,

$$B'C = B'B \cos BB'C;*$$

that is, $$r = 2R \cos v.$$

When v becomes equal to 90°, $\cos v = 0$, and r is 0. The radius-vector then becomes tangent to the circumference, since the two points in which it before cut it, have united.

From $v = 90°$, to $v = 270°$, the $\cos v$ is negative; and there is no point of the curve either in the second or third angle.

From $v = 270°$, to $v = 360°$, the $\cos v$ is positive, and the radius-vector will determine all the points of the semi-circumference, below the axis of abscissas.

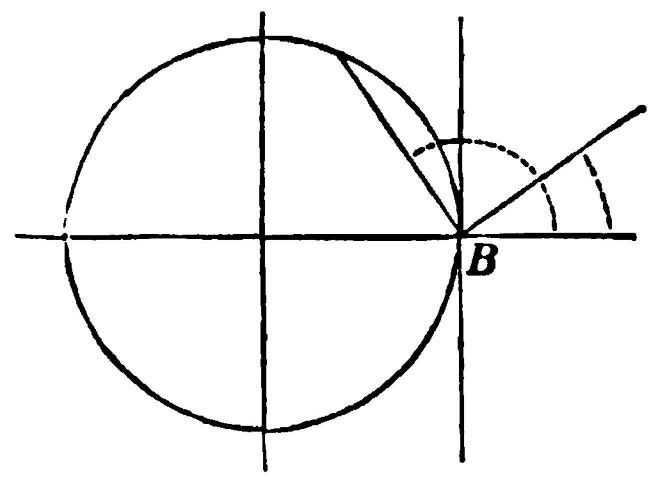
* Trig. Art. 30.

2. If the pole be placed at the point B, whose co-ordinates are,

$$a = +R, \qquad b = 0,$$

the equation will become,

$$r = -2R\cos v.$$



In this equation the radius-vector will be negative, when $\cos v$ is positive, and positive, when the $\cos v$ is negative.

Hence, the radius-vector will not give points of the curve from $v = 0$, to $v = 90°$. It will give points of the curve from $v = 90°$, to $v = 270°$; and it will again fail to determine a curve from $v = 270°$, to $v = 360°$. The figure verifies these results.

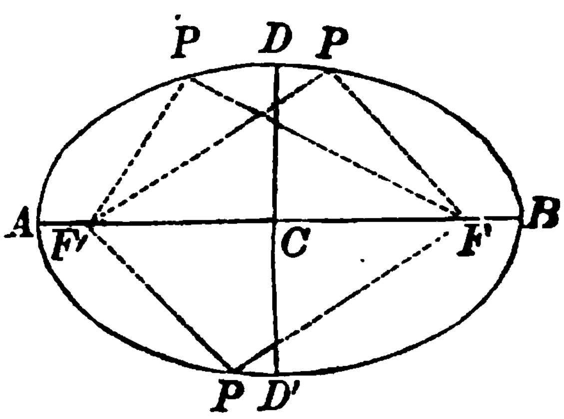
3. If we place the pole at the centre, the equations for transformation, will become,

$$x = r\cos v, \qquad y = r\sin v.$$

BOOK III.

OF THE ELLIPSE.

1. An Ellipse is a plane curve, such, that the sum of the two distances from any point, to two fixed points, is equal to a given distance. Thus, if F' and F be two fixed points, and AB a given distance; then, if $F'P + PF$, is constantly equal to AB, for every position of the point P, the curve $APBP$ will be an ellipse.



1. The fixed points, F' and F, are called *foci* of the ellipse.

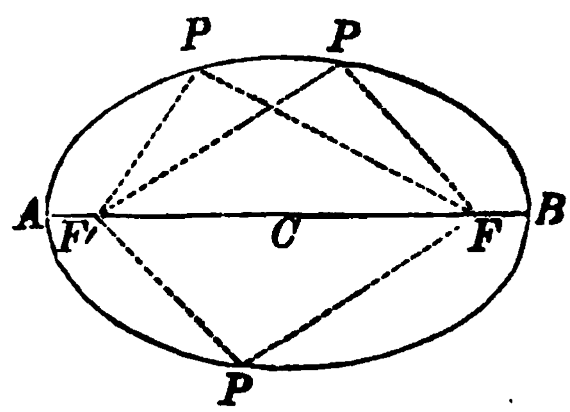
2. The line AB, passing through the foci, and limited by the curve, is called the *transverse axis;* and the extremities A and B, the *vertices* of the transverse axis.

3. The point C, on the transverse axis, and equally distant from the foci F' and F, is called the *centre* of the ellipse.

4. The line DD' drawn through the centre, perpendicular to the transverse axis, and limited by the curve, is called the *conjugate axis*, and D and D' are its vertices.

Construction of the Ellipse.

2.—1. We can easily construct an ellipse when its transverse axis and foci are given.



Let F' and F be the foci, and AB the transverse axis. Take a thread, equal in length to AB, and fasten its two extremities, the one at F', and the other at F. Press a pencil against the thread, and move it around the points F, F', keeping the thread constantly stretched: the point of the pencil will describe an ellipse; for, in every position of the pencil, we shall have,

$$F'P + PF = AB,$$

which is the characteristic property of the curve.

2. When the pencil is at B, we have,

$$AB = BF' + BF; \text{ but, } BF' = FF' + FB; \text{ hence,}$$

$$AB = FF' + 2BF \quad . \quad . \quad . \quad . \quad (1.)$$

When the pencil is at A, we have,

$$AB = AF + AF'; \text{ but, } AF = FF' + AF'; \text{ hence,}$$

$$AB = FF' + 2AF' \quad . \quad . \quad . \quad . \quad (2.)$$

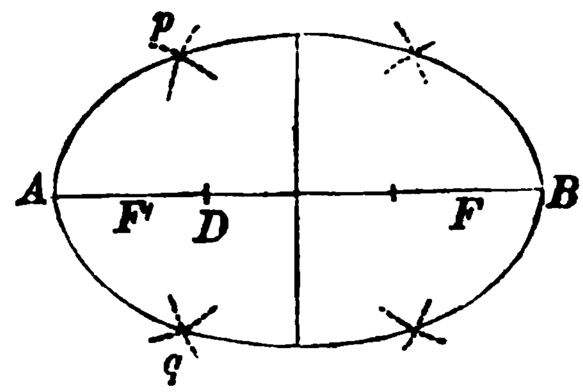
Equating the second members of Equations (1) and (2), cancelling the common term FF', and dividing by 2, we have,

$$BF = AF';$$

that is, *the distances from the foci to the vertices of the transverse axis are equal.*

Since the centre C is the middle point of $F'F$ (Art. 1–3), it follows, that the *centre C is also the middle point of the transverse axis.*

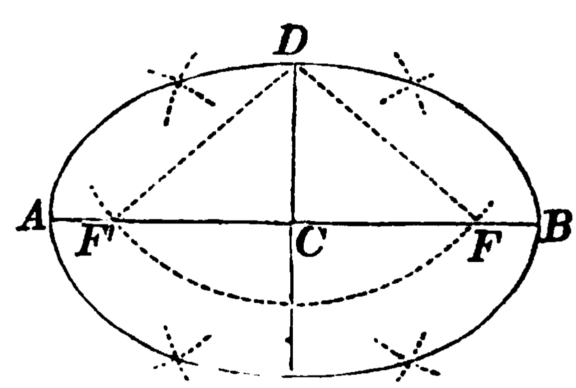
3. We may also construct the ellipse by *points*, when the transverse axis and foci are given.



Let AB be the transverse axis of an ellipse, and F' and F the foci. Take, in the dividers, any portion of the transverse axis, as AD, and with the focus F', as a centre, describe the arcs p and q. With BD, the remaining part of the transverse axis, as a radius, and the other focus F, as a centre, describe two other arcs intersecting the former; the points of intersection will be points of the curve. For, the sum of the distances from p or q, to F' and F, is equal to AB.

If with the radius AD, two arcs be described from the focus F, and with the radius BD two arcs be described from the focus F', these arcs will also determine, by their intersections, two points of the curve; so that, for each time we take a part AD of the transverse axis, we shall determine four points of the curve.

4. Construct an ellipse when its axes are given.



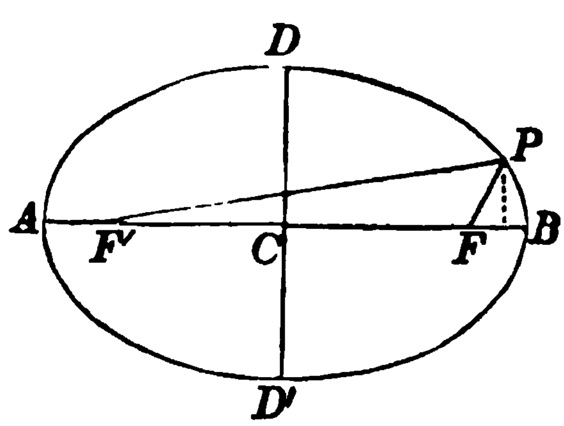
If from either vertex of the conjugate axis, as D, the lines DF', DF, be drawn to the foci, they will be equal to each other.

For, in the two right-angled triangles, $F'CD$, FCD, CF' is equal to CF, and CD is common; hence, the hypothenuse DF' is equal to DF.*

* Legendre, Bk. IV. Prop. 11.

But $F'D + DF$ is equal to AB; hence, DF', or DF, is equal to CB. If, therefore, with either vertex of the conjugate axis as a centre, and with a radius equal to half the transverse axis, the circumference of a circle be described, it will intersect the transverse axis at the foci. Having found the foci, the ellipse may be constructed, as in the last case.

Equation of the ellipse.



3. Let F' and F be the foci, and denote the distance between them by $2c$; then, CF or CF' $= c$.

Let P be any point of the curve. Designate the distance $F'P$, by r', and FP, by r. Let $2A = AB$, denote the the given line, to which the sum, $F'P + FP$, is to be constantly equal.

Through C, the middle point of $F'F$, draw CD perpendicular to $F'F$, and let C be the origin of a system of rectangular co-ordinates, of which AB, DD', are the axes. Denote the distance CD by B, and let x and y denote the co-ordinates of any point, as P.

The square of the distance between any two points, of which the co-ordinates are x, y, and x', y' (Bk. I., Art. **19**), is,

$$(y - y')^2 + (x - x')^2.$$

If the distance be estimated from the point F', of which the co-ordinates are $y' = 0$, and $x' = -c$, we shall have,

$$\overline{F'P}^2 = r'^2 = y^2 + (x + c)^2 \quad . \quad . \quad . \quad (1.)$$

If it be estimated from the point F, of which the co-ordinates are $y' = 0$, and $x' = + c$, we shall have,

$$r^2 = y^2 + (x - c)^2 \quad . \quad . \quad . \quad . \quad (2.)$$

Since the lines intersect each other, the co-ordinates of P will satisfy Equations (1) and (2); hence, the equations are simultaneous.

If we add and subtract them, we obtain,

$$r'^2 + r^2 = 2(y^2 + x^2 + c^2) \quad . \quad . \quad . \quad (3.)$$

and

$$r'^2 - r^2 = 4cx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Equation (4) may be placed under the form,

$$(r' + r)(r' - r) = 4cx \quad . \quad . \quad . \quad (5.)$$

But we have, from the property of the ellipse,

$$r' + r = 2A \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

Combining (5) and (6), we have,

$$r' - r = \frac{2cx}{A} \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

Combining (6) and (7), by addition and subtraction, we obtain,

$$r' = A + \frac{cx}{A} \quad . \quad . \quad (8.) \qquad \text{and} \qquad r = A - \frac{cx}{A} \quad . \quad . \quad (9.)$$

Squaring both members of Equations (8) and (9), combining the resulting equations, and substituting the values of r'^2 and r^2, in Equation (3), we obtain,

$$A^2 + \frac{c^2x^2}{A^2} = y^2 + x^2 + c^2.$$

Substituting for c^2, its value, $A^2 - B^2$, (Art. 2—4), we have,

$$A^2 + \frac{A^2 - B^2}{A^2} x^2 = y^2 + x^2 + A^2 - B^2;$$

or, $$A^4 + A^2x^2 - B^2x^2 = A^2y^2 + A^2x^2 + A^4 - A^2B^2.$$

Cancelling and transposing, we have,

$$A^2y^2 + B^2x^2 = A^2B^2,$$

which is the equation of the ellipse, referred to its centre and axes.

Interpretation of the equation.

4. 1. If, in the equation of the ellipse,

$$A^2y^2 + B^2x^2 = A^2B^2,$$

we make $y = 0$, the corresponding values of x will be the abscissas of the points in which the curve intersects the axis of X (Bk. II., Art. 4—1), viz.:

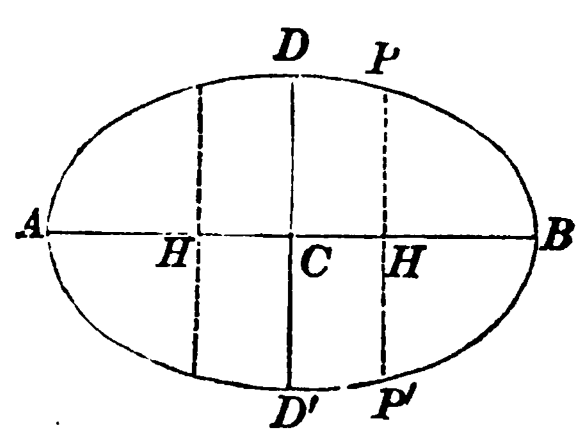
$$x = +A, \text{ for } B, \quad \text{and} \quad x = -A, \text{ for } A.$$

2. If we make $x = 0$, the corresponding values of y will be the ordinates of the points in which the curve intersects the axis of Y; viz.:

$$y = +B, \text{ for } D, \quad \text{and} \quad y = -B, \text{ for } D'.$$

3. If we place the equation of the ellipse under the form,

$$y = \pm \frac{B}{A}\sqrt{A^2 - x^2},$$

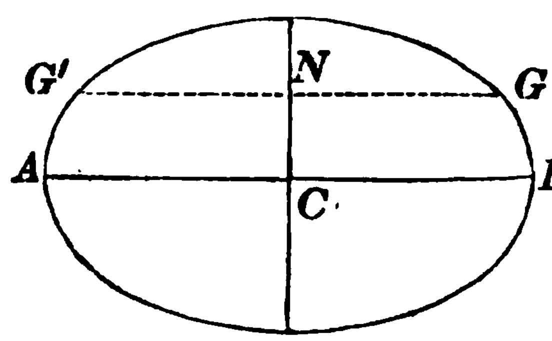


we see, that for every value of x, as CH, whether plus or

minus, there will be two values for y, numerically equal, with contrary signs; hence, the curve is symmetrical with respect to the transverse axis.

If x be made greater than A, whether it be taken plus or minus, the values of y will be imaginary; hence, the curve will be limited both in the direction of x positive and x negative.

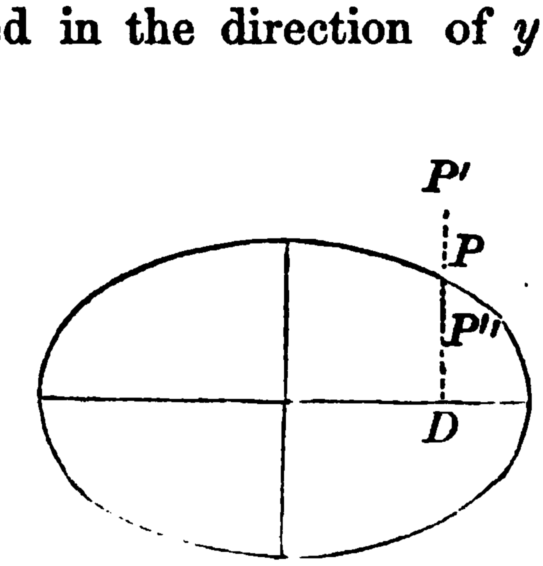
4. If we place the equation under the form,



$$x = \pm \frac{A}{B}\sqrt{B^2 - y^2},$$

we see, that for every value of y, whether positive or negative, as CN, there will be two equal values of x, with contrary signs; hence, the curve will be symmetrical with respect to the conjugate axis. If y be made greater than B, either positive or negative, the values of x will be imaginary; hence, the curve will be limited in the direction of y positive and y negative.

5. The equation of the ellipse,



$$A^2y^2 + B^2x^2 = A^2B^2,$$

may be put under the form,

$$A^2y^2 + B^2x^2 - A^2B^2 = 0,$$

and this equation will be satisfied, so long as x and y denote the co-ordinates of points of the curve.

If we take any point, as P', without the curve, its ordinate $P'D$, will be greater than the ordinate of the curve.

If we denote this ordinate by y, the first member of the equation, instead of reducing to 0, will be equal to a positive quantity.

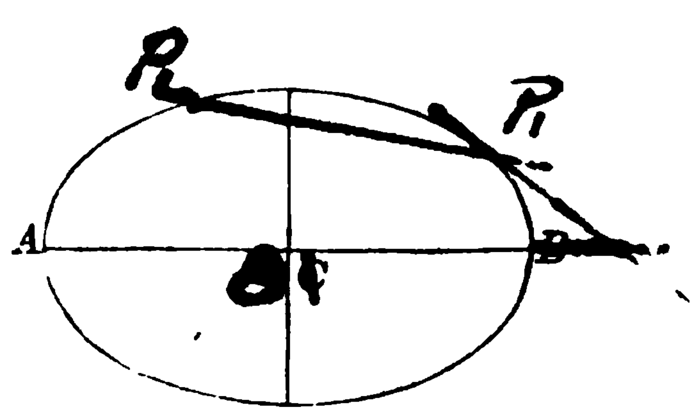
If, on the contrary, we take a point P'', within the curve, its ordinate $P''D$, will be less than the ordinate of the curve; and if we designate this line by y, the first member of the last equation will be negative.

Therefore, the following analytical conditions will determine the position of a point, with respect to the curve of the ellipse, viz.:

Without the curve,	$A^2y^2 + B^2x^2 - A^2B^2 > 0.$
In the curve,	$A^2y^2 + B^2x^2 - A^2B^2 = 0.$
Within the curve,	$A^2y^2 + B^2x^2 - A^2B^2 < 0.$

Equation when the origin is at the vertex of the transverse axis.

5. If we transfer the origin of co-ordinates from the centre C, to A, one extremity of the transverse axis, the equations of transformation (Bk. I., Art. **28**), will reduce to,



$$x = -A + x', \qquad y = y'.$$

Substituting these values in the equation of the ellipse, it reduces to,

$$A^2y'^2 + B^2x'^2 - 2B^2Ax' = 0,$$

which may be put under the form,

$$y'^2 = \frac{B^2}{A^2}(2Ax' - x'^2), \qquad \text{or,} \qquad y^2 = \frac{B^2}{A^2}(2Ax - x^2)$$

by omitting the accents. This is the equation of the ellipse referred to the vertex of the transverse axis, as an origin of co-ordinates. In this equation, the absolute term is wanting, as it should be, since the origin of co-ordinates is in the curve (Bk. II., Art. 4—9).

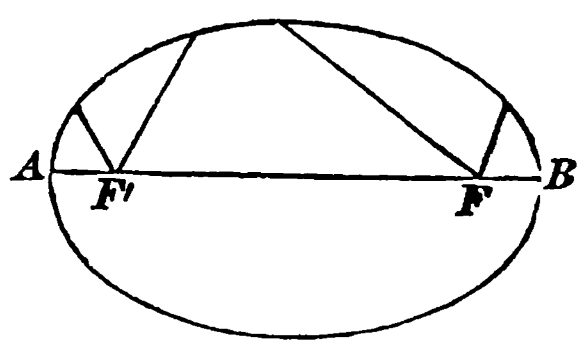
Eccentricity. Polar equation.

6. The ECCENTRICITY of an ellipse, is the distance from the centre to either focus, divided by the semi-transverse axis. If c denotes the distance from the centre to either focus (Art. **3**), and e the eccentricity, then,

$$\frac{\sqrt{A^2 - B^2}}{A} = \frac{c}{A} = e, \quad \text{and} \quad c = Ae \quad . \quad . \quad (1.)$$

1. Resuming Equations (8) and (9) (Art. **3**), we have,

$$r' = A + ex \quad . \quad . \quad (2.) \qquad \text{and} \qquad r = A - ex \quad . \quad . \quad (3.)$$



For the value of r', the pole is at F'; for r, it is at F, and the origin of co-ordinates is at the centre of the ellipse.

Let us transfer the origin of co-ordinates to the focus F'. For this point we have (Bk. I., Art. **28**),

$$x = -Ae + x', \quad \text{and} \quad y = y'.$$

Substituting this value of x in Equation (2), we have,

$$r' = A - Ae^2 + ex'.$$

But $x' = r' \cos v$ (Bk. I., Art. **35**). Substituting this value of x', we have,

$$r' = A - Ae^2 + er' \cos v;$$

whence, $$r' = \frac{A(1 - e^2)}{1 - e \cos v}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

which is the polar equation, when the pole is at F'.

2. If $v = 0$, $\cos v = 1$, and we have,

$$r' = \frac{A(1 - e^2)}{1 - e} = A(1 + e) = A + c = F'B.$$

If v increases from 0 to 360°, the corresponding values of r', will give all the points of the curve.

3. If we had transferred the origin of co-ordinates to the focus F, we should have had (Bk. I., Art. 28),

$$x = Ae + x', \qquad \text{and} \qquad y = y'.$$

Substituting this value of x in Equation (3), we have,

$$r = A - Ae^2 - ex'.$$

But $x' = r \cos v$. Substituting this value of x', we have,

$$r = A - Ae^2 - er \cos v;$$

whence, $$r = \frac{A(1 - e^2)}{1 + e \cos v}. \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

4. We see, from Equations (2) and (3), that when the pole is at either focus, the radius-vectors will be expressed in *rational functions* of the abscissas of the points in which they intersect the curve. It may be easily shown, that the foci are the *only points in the plane of the curve*, which enjoy this remarkable property.

5. If $v = 0$, $\cos v = 1$, and we have,

$$r = \frac{A(1 - e^2)}{1 + e} = A(1 - e) = A - c = FB.$$

If v varies from 0 to 360°, the corresponding values of r will give all the points of the curve. The difference between Equations (4) and (5) is this: for $v = 0$, the value of r' begins at the *remote vertex;* while in Equation (5), under the same supposition, the corresponding value of r, begins at the *nearest vertex.*

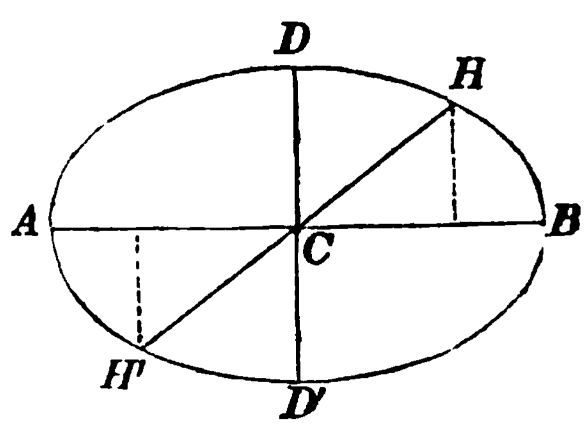
Diameters.

7. A DIAMETER of an ellipse, is a line drawn through the centre, and limited by the curve. The points in which it intersects the curve, are called *vertices* of the diameter.

Every diameter is bisected at the centre.

8. The equation of the ellipse, referred to its centre and axes, is,

$$A^2y^2 + B^2x^2 = A^2B^2 \quad . \quad . \quad . \quad . \quad (1.)$$



Since every diameter, as $H'CH$, passes through the origin of co-ordinates, its equation will be,

$$y = ax \quad . \quad . \quad (2.)$$

If Equations (1) and (2) be combined, the values of x and y, in the resulting equation, will be the co-ordinates of the vertices H and H'.

Combining and eliminating, we have,

$$x = \pm AB\sqrt{\frac{1}{A^2a^2 + B^2}}, \qquad y = \pm ABa\sqrt{\frac{1}{A^2a^2 + B^2}}.$$

If we denote the co-ordinates of the point H, by x', y', and the co-ordinates of H', by x'', y'', we shall have,

$$x' = AB\sqrt{\frac{1}{A^2a^2 + B^2}}, \qquad y' = ABa\sqrt{\frac{1}{A^2a^2 + B^2}}.$$

$$x'' = -AB\sqrt{\frac{1}{A^2a^2 + B^2}}, \quad y'' = -ABa\sqrt{\frac{1}{A^2a^2 + B^2}}.$$

Since the co-ordinates of these points are the same, with contrary signs, it follows that,

$$CH = CH';$$

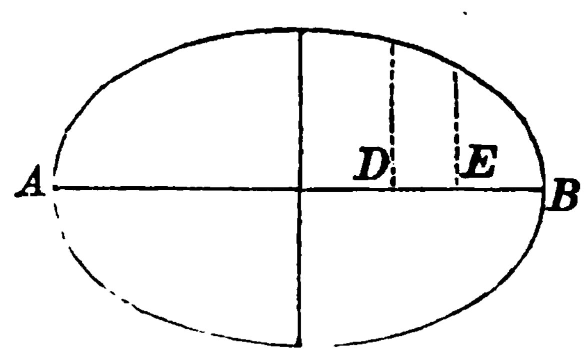
that is: *Every diameter is bisected at the centre.*

Ordinates to diameters.

9. An ordinate of a point, *to a diameter*, is its distance from the diameter, measured on a line parallel to a tangent at the vertex of the diameter. The parts into which the ordinate divides the diameter, are called *segments.*

Relation of ordinates to each other.

10. The equation of the ellipse referred to the vertex A, as the origin of co-ordinates (Art. **5**), is,



$$y^2 = \frac{B^2}{A^2}(2A - x)x.$$

If we designate a particular ordinate by y', and its abscissa by x'; and a second ordinate by y'', and its abscissa by x'', we shall have,

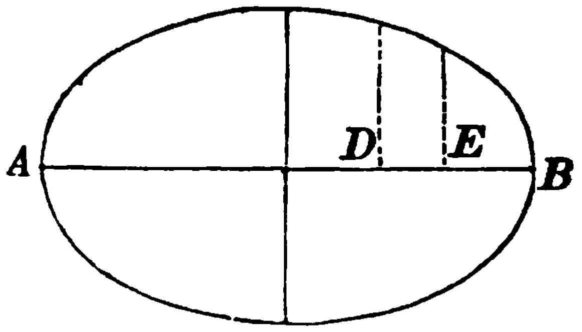
$$y'^2 = \frac{B^2}{A^2}(2A - x')x', \quad . \quad . \quad . \quad . \quad (1.)$$

and,
$$y''^2 = \frac{B^2}{A^2}(2A - x'')x'' \quad . \quad . \quad . \quad . \quad (2.)$$

Dividing Equation (1) by (2), we obtain,

$$\frac{y'^2}{y''^2} = \frac{(2A - x')x'}{(2A - x'')x''};$$

or, $\quad y'^2 : y''^2 :: (2A - x')x' : (2A - x'')x''.$

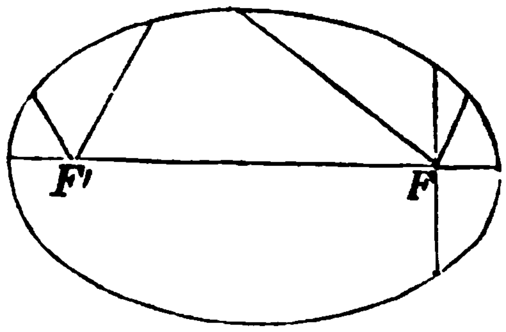


But $2A$ denotes the transverse axis AB, and since $x' = AD$, $2A - x' = DB$; therefore, $(2A - x')x'$, is the rectangle of the segments AD, DB. In like manner, $(2A - x'')x''$, is the product of the segments AE, EB. Since the same may be shown for the conjugate axis, we conclude that,

The squares of the ordinates, to either axis of the ellipse, are to each other as the rectangles of the corresponding segments into which they divide the axis.

Parameter.

11. The PARAMETER of the transverse axis, is the double ordinate passing through the focus.



To find its value, let us take the Polar Equation (5) (Art. 6), viz.:

$$r = \frac{A(1 - e^2)}{1 + e\cos v}.$$

If we make $v = 90°$, the radius-vector will be perpendicular to the transverse axis, and r will be equal to the ordinate. Under this supposition, $\cos v = 0$, and we shall have,

$$r = A(1 - e^2).$$

In Art. 6 we have,

$$e^2 = \frac{A^2 - B^2}{A^2}.$$

Substituting this value for e^2, we have,

$$r = A\left(1 - \frac{A^2 - B^2}{A^2}\right) = \frac{B^2}{A}.$$

Hence, $\quad$ parameter $= \dfrac{2B^2}{A} = \dfrac{4B^2}{2A} = 2\,r$

If we write this in a proportion, we have,

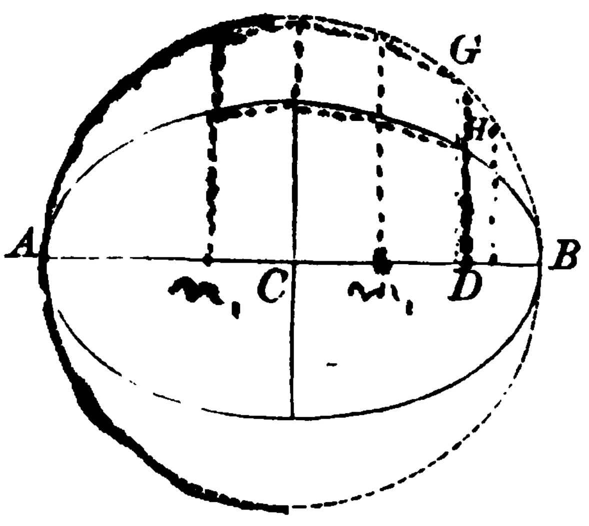
$2A \;:\; 2B \;::\; 2B \;:$ parameter; that is,

The parameter of the transverse axis is a third proportional to the transverse axis and its conjugate.

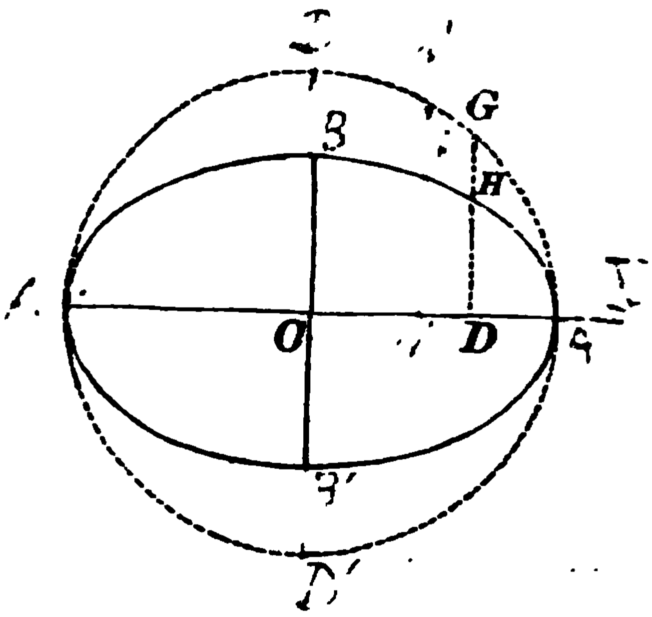
1. In the polar equation, when the pole is at the focus, the *numerator, in the value of r, is* equal to *half the parameter.*

Ellipse and circumscribing circle.

12. If on the transverse axis AB, the circumference of a circle be described, it is required to find the relation between any ordinate GD, of the circle, and the ordinate HD, of the ellipse, corresponding to the same abscissa CD.



Let Y' denote any ordinate of the circle, and y' the corresponding ordinate of the ellipse, and x' the common abscissa. We shall then have (Bk. II. Art. 4—5),



$$Y'^2 = A^2 - x'^2 \quad . \quad . \quad (1.)$$

We also have from the equation of the ellipse referred to its centre and axes (Art. 3),

$$y'^2 = \frac{B^2(A^2 - x'^2)}{A^2} \quad . \quad . \quad . \quad . \quad (2.)$$

Dividing Equation (2) by (1), member by member, we have

$$\frac{y'^2}{Y'^2} = \frac{B^2}{A^2}, \quad \text{or} \quad \frac{y'}{Y'} = \frac{B}{A};$$

Therefore, $\quad Y' : y' :: A : B$; that is,

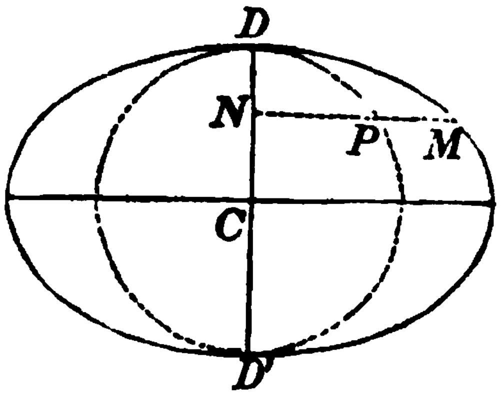
The ordinate of the circle, is to the corresponding ordinate of the ellipse, as the semi-transverse axis, to the semi-conjugate.

1. It follows, from the above proportion, that every ordinate of the circle is greater than the corresponding ordinate of the ellipse: hence, every point of the ellipse, except the vertices of the transverse axis, is within the circumference of the circle: therefore, *the transverse axis is greater than any other diameter.*

2. If B is made equal to A, the ellipse becomes the circumference of the circle described on the transverse axis.

Ellipse and inscribed circle.

13. If, on the conjugate axis, DD', the circumference of a circle be described, it is required to find the relation between any ordinate, PN, of the circle, and the ordinate MN, of the ellipse, corresponding to the same abscissa CN.



Denote the ordinates, NP and NM, by X' and x', and designate CN, by y'. We shall then have,

$$X'^2 = B^2 - y'^2,$$

and,

$$x'^2 = \frac{A^2}{B^2}(B^2 - y'^2);$$

hence,

$$\frac{x'^2}{X'^2} = \frac{A^2}{B^2}, \quad \text{or} \quad \frac{x'}{X'} = \frac{A}{B};$$

therefore,

$$X' : x' :: B : A; \quad \text{that is,}$$

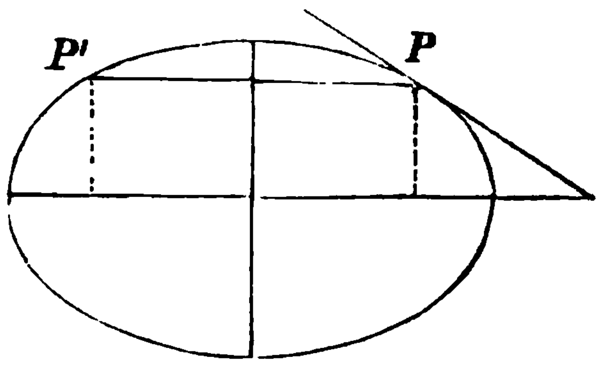
The ordinate of the circle is to the corresponding ordinate of the ellipse, as the semi-conjugate axis is to the semi-transverse.

1. Since $A > B$, every ordinate of the ellipse is greater than the corresponding ordinate of the circle: therefore, every point of the ellipse, except the vertices of the conjugate axis, is without the circumference of the circle. Hence, *the conjugate axis is less than any other diameter.*

2. If A is made equal to B, the ellipse becomes the circumference of the circle described on the conjugate axis.

Equation of the Tangent.

14. It is required to find the equation of a tangent line to the ellipse.



Take any point of the curve, as P, and designate its co-ordinates by x'', y''. Through this point, draw a secant line; its equation will be of the form,

$$y - y'' = a(x - x'') \quad . \quad . \quad . \quad . \quad (1.)$$

It is now required to find the value of a, when the secant line PP' becomes tangent to the curve.

The equation of the ellipse is,

$$A^2y^2 + B^2x^2 = A^2B^2 \quad . \quad . \quad . \quad . \quad (2.)$$

Since the point P is in the curve, we shall have,

$$A^2y''^2 + B^2x''^2 = A^2B^2 \quad . \quad . \quad . \quad . \quad (3.)$$

Subtracting (3) from (2), we have,

$$A^2(y^2 - y''^2) + B^2(x^2 - x''^2) = 0;$$

or,

$$A^2(y + y'')(y - y'') + B^2(x + x'')(x - x'') = 0. \quad (4.)$$

In this equation, x and y are the co-ordinates of any point of the ellipse.

If Equation (4), be combined with Equation (1), the co-ordinates x and y, in the resulting equation, will be the co-ordinates of the points in which the secant intersects the ellipse. These equations are most readily combined, by substituting for $y - y''$, in Equation (4), the value in Equation (1). Substituting, we obtain,

$$A^2(y + y'')a(x - x'') + B^2(x + x'')(x - x'') = 0;$$

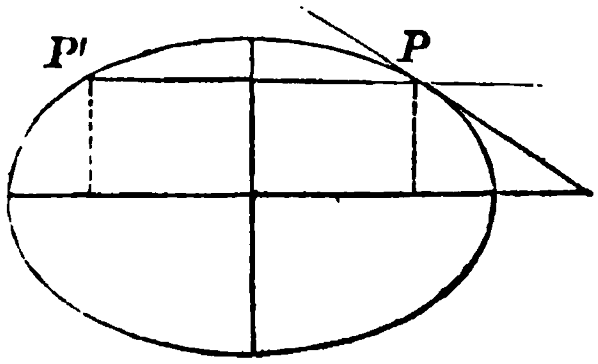
or, by factoring, we have,

$$(x - x'') \times [A^2a(y + y'') + B^2(x + x'')] = 0,$$

an equation, which may be satisfied by making,

$$x - x'' = 0; \quad \text{or,} \quad A^2a(y + y'') + B^2(x + x'') = 0.$$

In the first equation, x is the abscissa of P; in the second, x and y are the co-ordinates of P'.



If we suppose P' to move towards P, we shall have, when it coincides with it,

$$x = x'', \quad \text{and} \quad y = y'';$$

which will give, from the last equation,

$$a = -\frac{B^2x''}{A^2y''}.$$

Substituting this value in Equation (1), we have,

$$y - y'' = -\frac{B^2x''}{A^2y''}(x - x'');$$

or, by reducing,

$$A^2yy'' - A^2y''^2 = -B^2xx'' + B^2x''^2;$$

or, $$A^2yy'' + B^2xx'' = A^2y''^2 + B^2x''^2;$$

or, $$A^2yy'' + B^2xx'' = A^2B^2,$$

which is the equation of the tangent line, and in which, y and x are the general co-ordinates of its points.

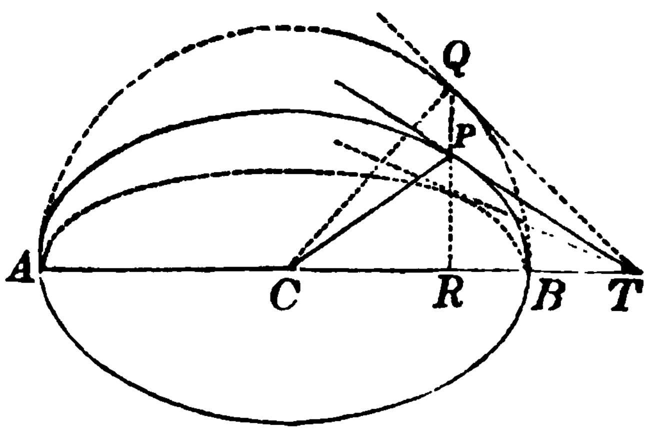
Sub-tangent.

15. A SUB-TANGENT is the projection of the tangent on the axis of abscissas, or on the axis of ordinates; that is,

it is the part of either axis, from the point of intersection, to the foot of the ordinate.

1. To find the sub-tangent, take the equation of the tangent,

$$A^2yy'' + B^2xx'' = A^2B^2.$$



If, in this equation, we make $y = 0$, we find,

$$x = \frac{A^2}{x''},$$

which is the line CT.

If from CT, we subtract CR, which is designated by x'', we shall have the sub-tangent,

$$TR = \frac{A^2}{x''} - x'' = \frac{A^2 - x''^2}{x''}.$$

2. This expression for the sub-tangent TR, is independent of the conjugate axis, and will, therefore, be the same for all ellipses having the same transverse axis AB, and the points of tangency in the same perpendicular RP. Hence, if the circumference of a circle be described on the transverse axis, and the ordinate RP be produced till it meets the curve at Q, the tangent, at this point, will pass through the common point T.

3. If we determine, in like manner, the sub-tangent on the conjugate axis, it will be independent of the transverse axis.

Equation of the normal.

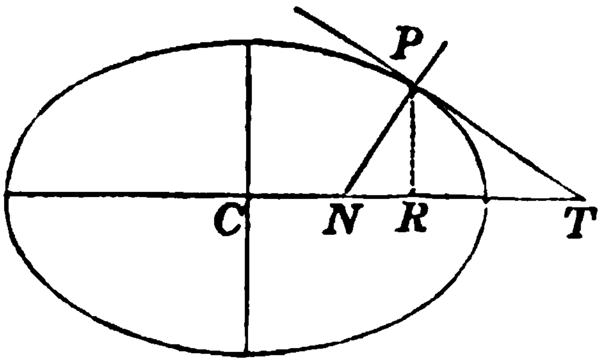
16. Since the normal passes through the point of tangency (Bk. II., Art. **9**), its equation will be of the form,

$$y - y'' = a'(x - x''), \quad . \quad . \quad . \quad (1.)$$

and since it is perpendicular to the tangent, we shall have,

$$aa' + 1 = 0.$$

But we have found (Art. **14**),



$$a = -\frac{B^2x''}{A^2y''};$$

hence, $$a' = \frac{A^2}{B^2}\frac{y''}{x''}.$$

Substituting this value in Equation (1), we have,

$$y - y'' = \frac{A^2}{B^2}\frac{y''}{x''}(x - x''),$$

which is the equation of the normal line.

Sub-normal.

17. A SUB-NORMAL is the projection of the normal on the axis; that is, it is the part of the axis which lies directly under the normal.

1. To find the sub-normal NR, take the equation of the normal,

$$y - y'' = \frac{A^2}{B^2}\frac{y''}{x''}(x - x''),$$

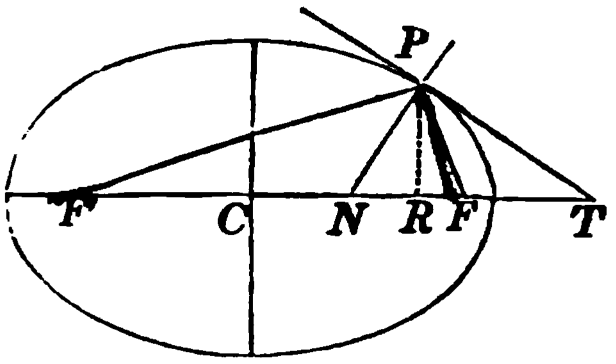
and make $y = 0$; this will give,

$$x = CN = \frac{A^2 - B^2}{A^2}x'' = e^2x'' \quad \text{(Art. 6)}.$$

If we subtract this value from CR, which is denoted by x'', we shall have the sub-normal,

$$NR = \frac{B^2x''}{A^2}.$$

Normal bisects the angle of the two lines drawn to the foci.



18. If from P, any point of the curve, we draw two lines to the foci F' and F, and recollect that CF', or CF, is equal to $c = Ae$ (Art. **6**), we have, by using the value of $CN = e^2x''$ (Art. **17**),

$$F'N = F'C + CN = Ae + e^2x'' = e(A + ex'');$$

and, $$FN = CF - CN = Ae - e^2x'' = e(A - ex'').$$

Hence, $$F'N : FN :: A + ex'' : A - ex''.$$

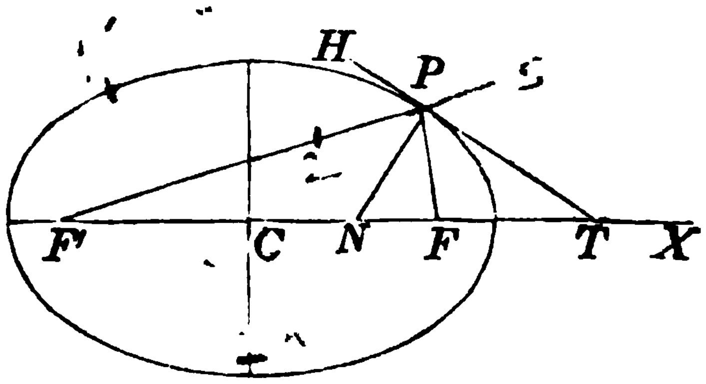
By referring to the values of r' and r (Art. **3**, Equations (8) and (9), and recollecting that $\frac{c}{A} = e$, we have,

$$r' : r :: A + ex : A - ex;$$

hence, $$r' : r :: F'N : FN;$$

therefore, PN bisects the angle $F'PF$.*

Tangent line and lines to the foci.



19. Let C be the centre of the ellipse, PT a tangent, and PF', PF, two lines drawn to the foci. Draw the normal, PN. Then, since NPH and NPT are right angles, they are equal. From each,

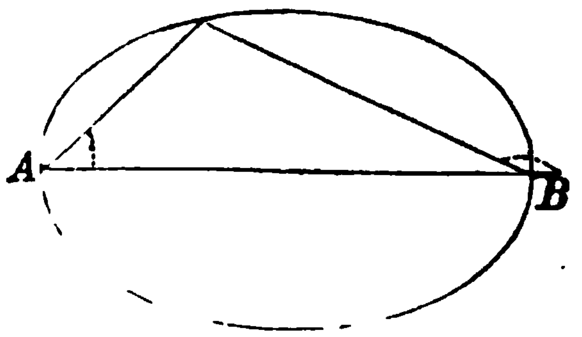
* Legendre, Bk. IV. Prop. 17.

take the equal angles, NPF' and NPF, and there will remain $F'PH$, equal to FPT. Hence,

If a line be drawn tangent to an ellipse at any point, and two lines be drawn from the same point to the two foci, the lines drawn to the foci will make equal angles with the tangent.

Supplementary chords.

20. Let AB be the transverse axis of an ellipse.



If a straight line be drawn through the point A, whose co-ordinates are,

$$x' = -A, \qquad y' = 0,$$

its equation will be,

$$y = a(x + A).$$

If a line be drawn through B, whose co-ordinates are,

$$x' = A, \qquad \text{and} \qquad y' = 0,$$

its equation will be,

$$y = a'(x - A).$$

If these lines intersect each other, we have,

$$y^2 = aa'(x^2 - A^2); \quad . \quad . \quad . \quad . \quad (1.)$$

and if they intersect on the curve of the ellipse, x and y must satisfy the equation,

$$y^2 = \frac{B^2}{A^2}(A^2 - x^2) = -\frac{B^2}{A^2}(x^2 - A^2). \quad . \quad . \quad (2.)$$

By combining Equations (1) and (2), we have,

$$aa' = -\frac{B^2}{A^2}; \text{ that is,}$$

If, through the vertices of the transverse axis, two supplementary chords be drawn, the product of the tangents of the angles which they form with it, will be negative, and equal to the square of the ratio of the semi-axes.

1. Since the product of the tangents is negative, the angles to which they correspond will fall in different quadrants.*

2. In the equation,

$$aa' = -\frac{B^2}{A^2},$$

there are two undetermined quantities, a and a'; hence, an infinite number of pairs of supplementary chords may be drawn through the extremities of the diameter AB.

If, however, a value be assigned to a, or a', that is, if one of the supplementary chords be given in position, the equation of condition will determine the other, and therefore, the corresponding supplementary chord may also be drawn.

3. If the ellipse becomes a circle, we shall have,

$$aa' = -1,$$

or,

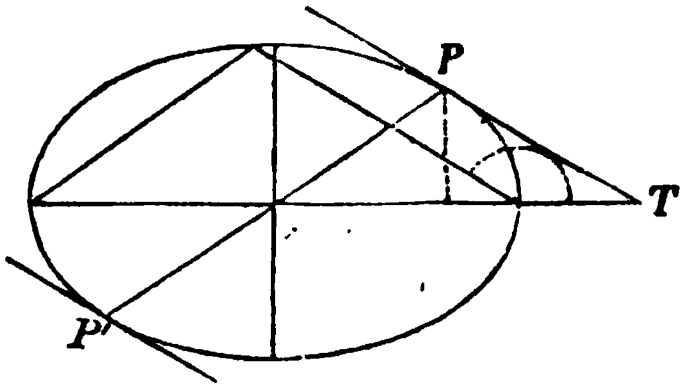
$$aa' + 1 = 0;$$

which shows, that the supplementary chords are perpendicular to each other, a property before proved (Bk. II., Art. 7).

* Legendre, Trig. Art. 18.

Supplementary chords. Tangent and diameter.

21. Let PT be a tangent line to the ellipse, and denote the co-ordinates of the point of contact by x'', y''. Then (Art. **14**),



$$a = -\frac{B^2x''}{A^2y''} \quad . \quad . \quad (1.)$$

If a diameter be drawn through the point P, its equation will be,

$$y'' = a'x'', \qquad \text{and} \qquad a' = \frac{y''}{x''} \quad . \quad . \quad (2.)$$

Multiplying Equations (1) and (2), member by member,

$$aa' = -\frac{B^2}{A^2}.$$

When a and a' denoted the tangents of the angles which the supplementary chords make with the transverse axis, we had (Art. **20**),

$$aa' = -\frac{B^2}{A^2};$$

hence, $$aa' = aa'.$$

If, in this equation, we make,

$$a = a,$$

we shall have, $$a' = a'; \quad \text{that is,}$$

If one chord is parallel to the tangent, the other will be parallel to the diameter passing through the point of contact.

Or, if we make, $a' = a',$

we shall have, $a = a$; that is,

If one of the chords be made parallel to the diameter, the other will be parallel to the tangent.

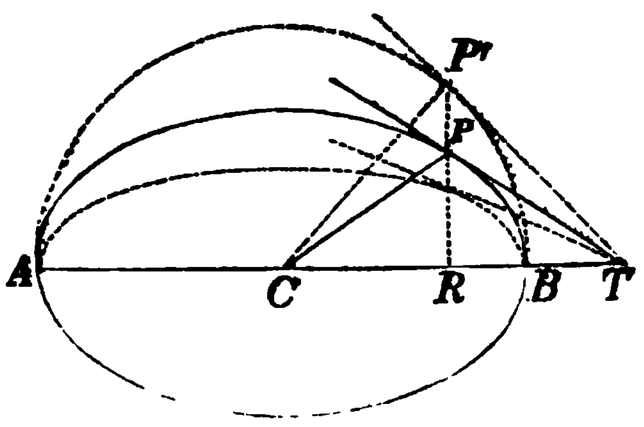
1. Since the co-ordinates of the points P and P', are the same with contrary signs (Art. 8), the value of a, in Equation (1), will be the same, whether we consider the tangent at P or P'; hence,

The tangents drawn through the extremities of the same diameter are parallel.

Construction of tangent lines to the ellipse.

22. Construct a tangent line to an ellipse, at a given point of the curve, when the axes are given.

First Method.



Let P be the given point. On the transverse axis AB, describe a semi-circumference, and through P, draw PR perpendicular to AB, and produce it till it meets the circumference at P'. Through P', draw a tangent line to the circumference of the circle, and from T, where it meets AB produced, draw TP, and it will be tangent to the ellipse at P (Art. 15).

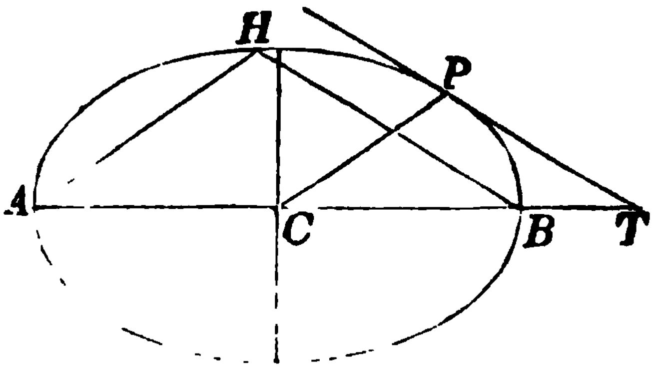
1. The angle $CP'T$ being a right angle, the angle CPT, which lies within it, is obtuse. Hence, the angle formed

by a tangent line, and the diameter passing through the point of contact, is, in general, obtuse.

If the point of tangency be at either vertex of the transverse axis, the tangent line to the ellipse will coincide with the tangent line to the circle, and will then be perpendicular to the transverse axis. Or, if the point of contact is at either vertex of the conjugate axis, the tangent line to the ellipse will become parallel to the tangent line to the circle, and, consequently, perpendicular to the conjugate axis.

Second method.

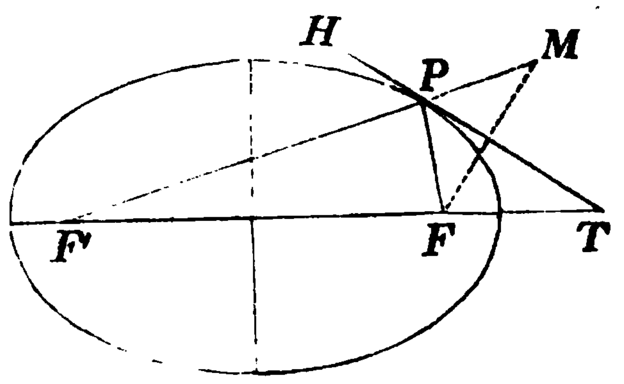
23. Let C be the centre of an ellipse, AB the transverse axis, and P the point of the curve at which the tangent is to be drawn.



Through P, draw the semi-diameter PC, and through A, draw the supplementary chord AH, parallel to it. Then draw the other supplementary chord BH, and through P, draw PT parallel to BH; then will PT be the tangent required (Art. **21**).

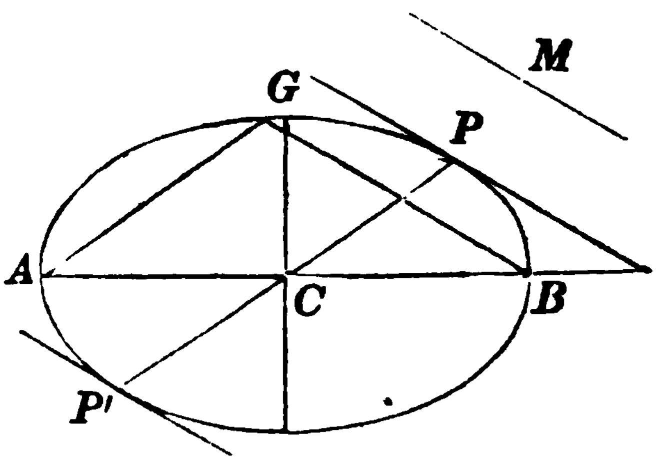
Third method.

24. Let P be the given point. Find the foci F' and F. (Art. **2—4**.) From P, draw the lines PF' and PF to the foci. Produce $F'P$, until PM shall be equal to PF, and draw FM. Then draw PT perpendicular to FM, and it will be the tangent required, since it makes equal angles with the lines PF' and PF (Art. **19**).



To draw a tangent parallel to a given line.

25. Let AB be the transverse axis, and M the given line.

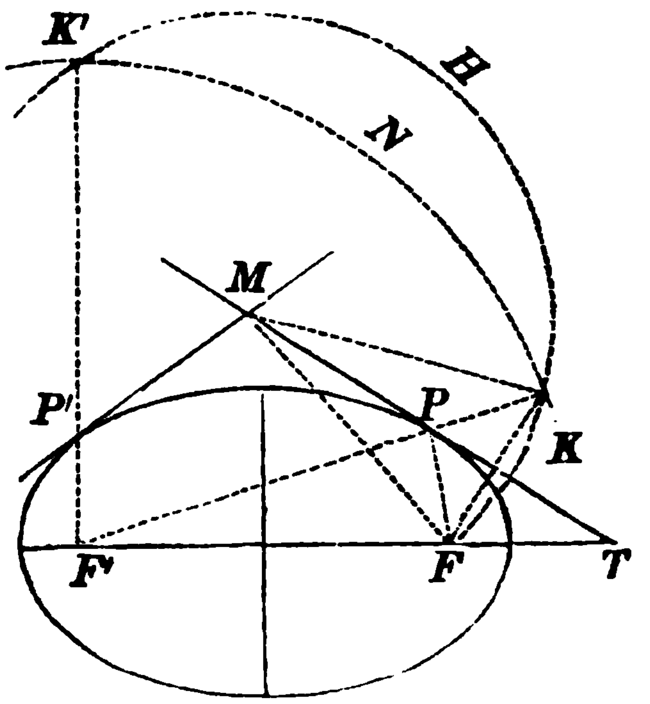


Through the vertex B draw the supplementary chord BG, parallel to M.

Then draw AG, and through the centre C draw CP parallel to AG, and produce it till it meets the ellipse again at P'. Through P, or P', draw a parallel to GB, and it will be the tangent required.

1. We see, from this construction, that if two tangents be drawn to the ellipse through the two extremities of the same diameter, they will be parallel to each other.

To draw a tangent through a point without the curve.



26. Let M be the given point. With either focus, as F', as a centre, and a radius equal to the transverse axis, describe the arc KNK'. Then, with M as a centre, and a radius equal to MF, the distance to the other focus, describe the arc $FKHK'$, intersecting the former in K and K'. Through K, draw KF'; and through P, where it

intersects the ellipse, draw the straight line MPT, and it will be tangent to the ellipse, at P.

For, since P is a point of the ellipse, $F'P + PF$ is equal to the transverse axis. But $F'P + PK$ is equal to the transverse axis, by construction. Hence, $PF = PK$.

Further, since the arc FK is described from the centre M, $MF = MK$; hence, the line MP has two of its points each equally distant from the points F and K; it is, therefore, perpendicular to FK*; and since the triangle FPK is isosceles, MT will bisect the vertical angle P. The opposite angle $F'PM$, being equal to TPK, is equal to FPT; hence, MT is tangent to the ellipse (Art **19**).

1. The two arcs KHK', KNK', will, in general, intersect each other in two points, K and K'. There will, therefore, be two lines, KF', $K'F'$, drawn to the focus F'; hence, there will be two points of contact, P, P', and, consequently, two tangent lines, MP, MP'.

CONJUGATE DIAMETERS.

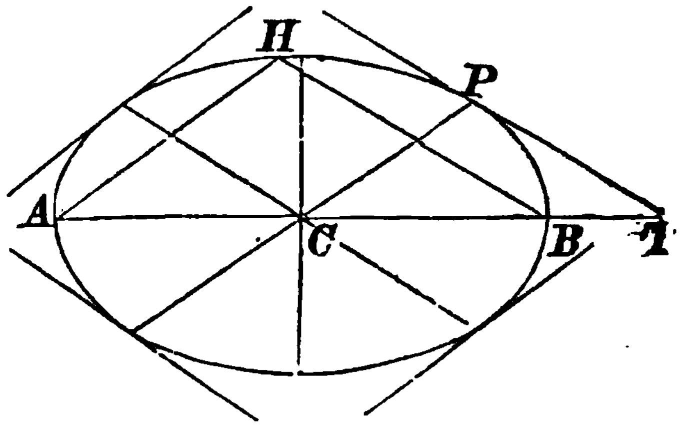
27. Two diameters of an ellipse are said to be conjugate to each other, when either of them is parallel to the two tangents drawn through the vertices of the other.

Since two supplementary chords may be drawn, respectively parallel to any diameter and the tangent through its vertex (Art. **21**), it follows, that two supplementary chords may always be drawn parallel to any two conjugate diameters.

If, therefore, we designate by a and a', the tangents of the angles which two conjugate diameters make, respectively, with the transverse axis, these tangents will

* Legendre, Bk. I. Prop. 16. Cor.

fulfill the condition of supplementary chords, and satisfy the equation,



$$aa' = -\frac{B^2}{A^2}.$$

Let us designate the corresponding angles by α and α'. We shall then have,

$$a = \frac{\sin\alpha}{\cos\alpha}, \quad \text{and} \quad a' = \frac{\sin\alpha'}{\cos\alpha'}.$$

Substituting these values in the last equation, and reducing, we obtain,

$$A^2 \sin\alpha \sin\alpha' + B^2 \cos\alpha \cos\alpha' = 0,$$

and dividing both members by $\cos\alpha \cos\alpha'$, we have,

$$A^2 \tan\alpha \tan\alpha' + B^2 = 0,$$

an equation, which expresses the relation between the angles which two conjugate diameters form with the transverse axis. It is called, *the equation of condition of conjugate diameters.*

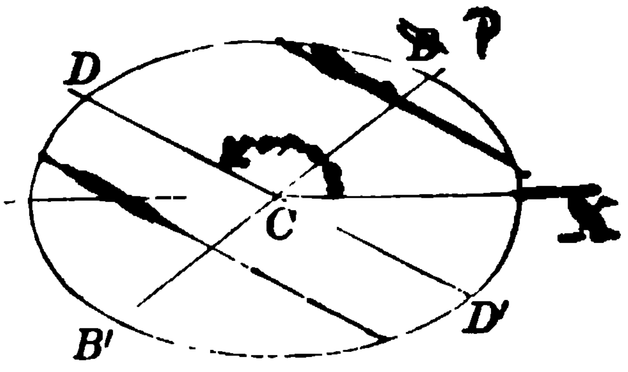
In the equation of condition, α and α' are undetermined. Hence, any value may be assigned to either of them; and when assigned, the value of the other can be determined from the equation of condition.

If $\alpha = 0$, we shall have, $\sin\alpha = 0$, and $\cos\alpha = 1$. Hence, $B^2 \cos\alpha' = 0$, and, consequently, $\cos\alpha' = 0$; or, $\alpha' = 90°$. Therefore, when one of the conjugate diameters coincides with the transverse axis, the other will coincide with the conjugate axis. The axes, therefore, fulfill the condition of conjugate diameters, as they should do, since each is parallel to the tangents drawn through the vertices of the other.

Ellipse referred to its centre and conjugate diameters.

28. The equation of the ellipse, referred to its centre and axes, is,

$$A^2y^2 + B^2x^2 = A^2B^2.$$



Let $B'B$ and DD', be two conjugate diameters.

It is required to refer the ellipse to these as a system of oblique axes, and to find its equation.

The formulas for passing from a system of rectangular to a system of oblique co-ordinates, the origin remaining the same (Bk. I., Art. **29**), are,

$$x = x' \cos\alpha + y' \cos\alpha', \qquad y = x' \sin\alpha + y' \sin\alpha'.$$

Squaring these values of x and y, and substituting in the equation of the ellipse, we obtain the equation of the curve, referred to conjugate diameters; viz.,

$$\left\{\begin{array}{c}(A^2\sin^2\alpha' + B^2\cos^2\alpha')y'^2 + (A^2\sin^2\alpha + B^2\cos^2\alpha)x'^2 \\ +2(A^2\sin\alpha\sin\alpha' + B^2\cos\alpha\cos\alpha')x'y'\end{array}\right\} = A^2B^2.$$

But the equation of condition, that the new co-ordinate axes shall be conjugate diameters (Art. **27**), is,

$$A^2\sin\alpha\sin\alpha' + B^2\cos\alpha\cos\alpha' = 0;$$

hence, the equation reduces to,

$$(A^2\sin^2\alpha' + B^2\cos^2\alpha')y'^2 + (A^2\sin^2\alpha + B^2\cos^2\alpha)x'^2 = A^2B^2.$$

To find the semi-diameter CB, make $y = 0$; then,

$$x'^2 = \frac{A^2B^2}{A^2\sin^2\alpha + B^2\cos^2\alpha} = CB^2 = CB'^2 = A'^2.$$

If we make $x' = 0$, we shall have,

$$y'^2 = \frac{A^2B^2}{A^2 \sin^2 \alpha' + B^2 \cos^2 \alpha'} = CD^2 = CD'^2 = B'^2.$$

The denominators, in the two last equations, are the coefficients of x'^2, y'^2, in the equation of the curve, referred to oblique axes. Finding their values, and substituting them in that equation, we have,

$$\frac{y'^2}{B'^2} + \frac{x'^2}{A'^2} = 1; \text{ hence,}$$

$$A'^2y'^2 + B'^2x'^2 = A'^2B'^2;$$

or, omitting the accents of x and y, since they are general variables,

$$A'^2y^2 + B'^2x^2 = A'^2B'^2.$$

which is the equation of the ellipse, referred to its centre and conjugate diameters.

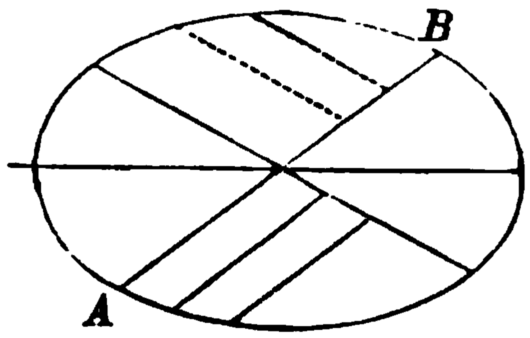
This equation, being of the same form as the equation of the ellipse, referred to its centre and axes, it follows, that every value of x will give two equal values of y, with contrary signs; and every value of y, two equal values of x, with contrary signs; hence, the ellipse is symmetrical with respect to either of its conjugate diameters; that is,

Either diameter bisects all chords drawn parallel to the other and terminated by the curve.

Relation of ordinates to each other.

29. The equation of the ellipse, referred to its conjugate diameters, is,

$$A'^2y^2 + B'^2x^2 = A'^2B'^2.$$



If we designate any two ordinates to the diameter AB, by y', y'', and the corresponding abscissas by x', x'', we shall have,

$$\frac{y'^2}{y''^2} = \frac{(A' + x')(A' - x')}{(A' + x'')(A' - x'')},$$

$$y'^2 : y''^2 :: (A' + x')(A' - x') : (A' + x'')(A' - x'').$$

If the ordinates be drawn to the conjugate diameter, it may be readily shown, that,

$$x'^2 : x''^2 :: (B' + y')(B' - y') : (B' + y'')(B' - y'')$$

Hence, *the squares of the ordinates to either of two conjugate diameters, are to each other as the rectangles of the segments into which they divide the diameter.*

Parameter.

30. The *Parameter* of any diameter is a third proportional to the diameter and its conjugate. Thus, if P designate the parameter of the diameter $2A'$, we shall have,

$$2A' : 2B' :: 2B' : P,$$

or, $$P = \frac{2B'^2}{A'}.$$

Relations between the axes and conjugate diameters.

31. The equation of the ellipse, referred to its conjugate diameters, which are oblique axes, is

$$A'^2y'^2 + B'^2x'^2 = A'^2B'^2.$$

It is required to refer the ellipse to its transverse and conjugate axes, which is a rectangular system.

The formulas for passing from oblique to rectangular axes, the origin remaining the same (Bk. I., Art. 30), are,

$$x' = \frac{x \sin \alpha' - y \cos \alpha'}{\sin (\alpha' - \alpha)} \qquad y' = \frac{y \cos \alpha - x \sin \alpha}{\sin (\alpha' - \alpha)}.$$

Substituting these values of y', x', we have,

(1.)

$$(A'^2 \cos^2\alpha + B'^2 \cos^2\alpha')y^2 + (A'^2 \sin^2\alpha + B'^2 \sin^2\alpha')x^2$$
$$-2(A'^2 \sin\alpha \cos\alpha + B'^2 \sin\alpha' \cos\alpha')xy = A'^2B'^2 \sin^2(\alpha' - \alpha),$$

which is the equation of the ellipse, referred to its centre and axes.

The form of the equation of the ellipse, referred to its axes, is

$$A^2y^2 + B^2x^2 = A^2B^2 \quad . \quad . \quad . \quad . \quad (2.)$$

hence, Equation (1) must be identical with Equation (2); that is, the coefficients of the like powers of the variables and the absolute terms, must be equal, each to each;* and since the product of the variables does not enter, in Equation (2), its coefficient, in Equation (1), must be zero. Hence, we have,

$$A'^2 \cos^2\alpha + B'^2 \cos^2\alpha' = A^2 \quad . \quad . \quad . \quad . \quad (1.)$$

$$A'^2 \sin^2\alpha + B'^2 \sin^2\alpha' = B^2 \quad . \quad . \quad . \quad . \quad (2.)$$

$$A'^2 \sin\alpha \cos\alpha + B'^2 \sin\alpha' \cos\alpha' = 0,$$

$$A'^2B'^2 \sin^2(\alpha' - \alpha) = A^2B^2,$$

or, $$A'B' \sin(\alpha' - \alpha) = AB \quad . \quad . \quad . \quad . \quad (3.)$$

* Bourdon, Art. 195. University, Art. 178.

The equation of condition, of conjugate diameters (Art. 27), is,

$$A^2 \tan \alpha \tan \alpha' + B^2 = 0.$$

If we add Equations (1) and (2), and recollect, that $\cos^2\alpha + \sin^2\alpha = 1$, we have,

$$A'^2 + B'^2 = A^2 + B^2.$$

Uniting Equation (3) with the last two equations, and changing the order, for convenience of interpretation, we have,

$$A^2 \tan \alpha \tan \alpha' + B^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1.)$$

$$A'B' \sin (\alpha' - \alpha) = AB \quad . \quad . \quad . \quad . \quad . \qquad (2.)$$

$$A^2 + B'^2 = A^2 + B^2 \quad . \quad . \quad . \qquad (3.)$$

These three equations express the relations that exist between α and α', the semi-axes, A and B, and any two semi-conjugate diameters, A' and B'.

Interpretation of $A^2 \tan \alpha \tan \alpha' + B^2 = 0$.

1. If we know the angle which the conjugate diameters make with each other, it will be equivalent to knowing α or α'. For, denote the known angle by β;

then, $\qquad \alpha' - \alpha = \beta;$

or, $\qquad \alpha' = \beta + \alpha; \qquad$ hence,

$$\tan \alpha' = \tan (\beta + \alpha) = \frac{\tan \beta + \tan \alpha}{1 - \tan \beta \tan \alpha}$$

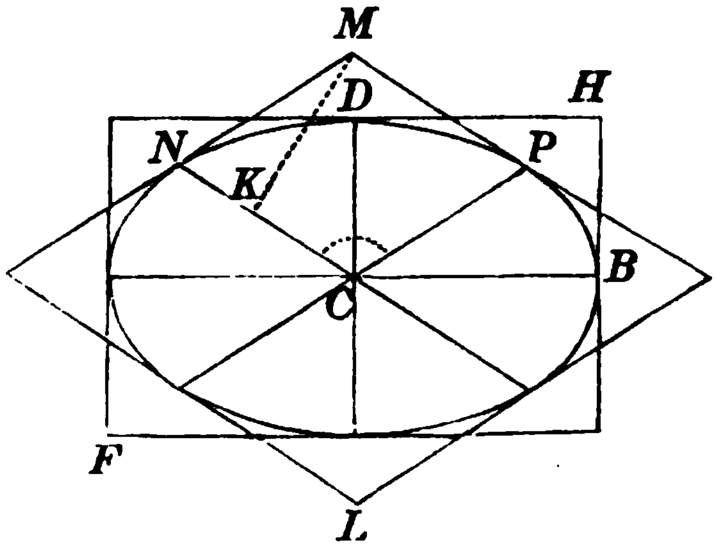
Substituting this value of $\tan \alpha'$ in Equation (1), we have,

$$A^2 \tan^2\alpha + (A^2 - B^2) \tan \alpha \tan \beta + B^2 = 0;$$

from which we can find $\tan\alpha$, and, consequently, α, in terms of the axes and the known quantity, $\tan\beta$.

Interpretation of $A'B' \sin(\alpha' - \alpha) = AB$.

2. Let us suppose the ellipse, whose centre is C, to be circumscribed by a rectangle, formed by drawing tangents at the vertices of the axes, and also by a parallelogram, formed by drawing tangents at the vertices of the conjugate diameters. Denote the semi-conjugates, CP and CN, by A' and B'.



From M, draw MK perpendicular to CN. The angle NCP is designated by $\alpha' - \alpha$, and since MNC is the supplement of NCP, its sine will be equal to $\sin(\alpha' - \alpha)$. Further, $NM = CP = A'$. Therefore,*

$$MK = A' \sin(\alpha' - \alpha).$$

Hence, $\qquad A'B' \sin(\alpha' - \alpha) = CPMN.$†

The second member of Equation (2), $A \times B$, is equivalent to the rectangle $CBHD$. But the parallelogram $CPMN$ is one-fourth the parallelogram ML, and the rectangle $CBHD$ is one-fourth the rectangle HF; hence, Equation (2) expresses the following property:

The rectangle which is formed by drawing tangents

* Legendre, Trig. Art. 32. † Mens. Art. 4.

through the vertices of the axes, is equivalent to the parallelogram which is formed by drawing tangents through the vertices of two conjugate diameters.

Interpretation of $A'^2 + B'^2 = A^2 + B^2$.

3. If we multiply both members by 4, we have,

$$4A'^2 + 4B'^2 = 4A^2 + 4B^2,$$

which expresses the following property

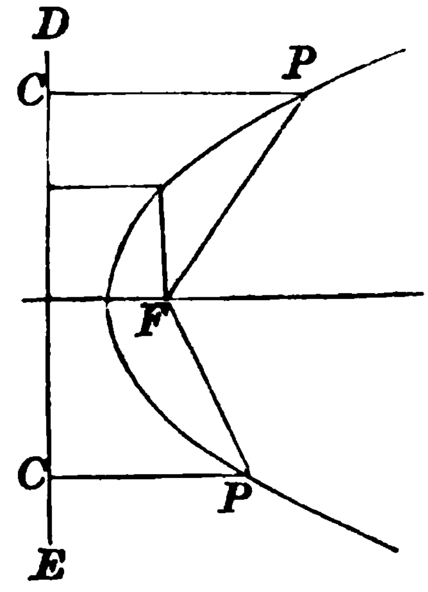
The sum of the squares described on the axes of an ellipse, is equivalent to the sum of the squares described on any two conjugate diameters.

The area of the ellipse is found in the Calculus, page 75.

BOOK IV.

OF THE PARABOLA.

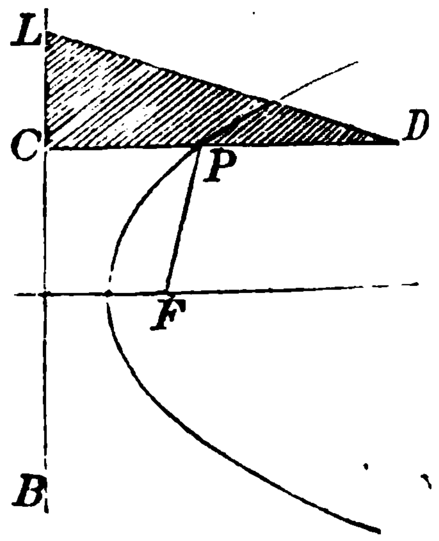
1. The Parabola is a plane curve, such that any point of it is equally distant from a fixed point and a given straight line.



The fixed point is called the *focus* of the parabola, and the given straight line, the *directrix*.

Thus, if F be a fixed point, and ED a given line, and the point P be so moved, that PF shall be constantly equal to PC, the point P will describe a parabola, of which F is the focus, and DE the directrix.

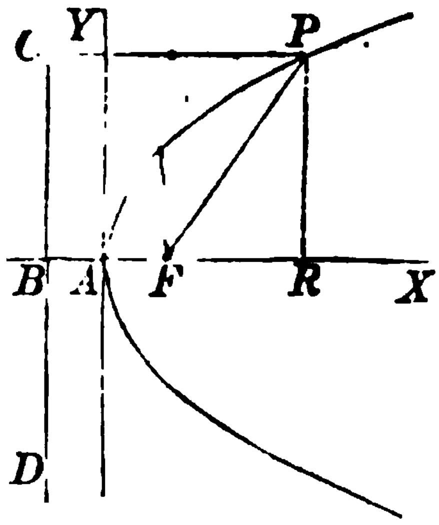
1. This property of the parabola affords an easy method of describing it mechanically.



Let BL be a given line, and LCD a triangular ruler, right-angled at C. Take a thread, the length of which is equal to the side CD, and attach one extremity at D, and the other at any point, as F. Place a pencil against the thread and the ruler, making tense the parts of the thread FP. PD. Then, if the side

CL of the ruler, be moved along the line BL, the pencil will describe a parabola, of which F is the focus, and BL the directrix; for, the distance PF will be equal to PC, for every position of the ruler.

Equation of the Parabola.



2. Let F be the focus, and DC the directrix. Denote the distance FB, from the focus to the directrix, by p, and let the point A, equally distant from B and F, be assumed as the origin of a system of rectangular co-ordinates, of which AX, AY, are the axes. The distance AF, will be denoted by $\frac{p}{2}$.

Let P be any point of the curve, and denote its co-ordinates by x and y.

Then, the distance between any two points (Bk. 1. Art. **19**), is,

$$\sqrt{(x'' - x')^2 + (y'' - y')^2}.$$

Substituting for x'', y'', the co-ordinates of the point P, and for x', y', the co-ordinates of F, which are, $x' = \frac{p}{2}$, and $y' = 0$, we have,

$$FP = \sqrt{y^2 + \left(x - \frac{p}{2}\right)^2}.$$

But, by the definition of the curve,

$$FP = PC = BA + AR = \frac{p}{2} + x \quad . \quad . \quad . \quad (1.)$$

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Hence, $$\sqrt{y^2 + \left(x - \frac{p}{2}\right)^2} = \frac{p}{2} + x,$$

or, $$y^2 + x^2 - px + \frac{p^2}{4} = \frac{p^2}{4} + px + x^2;$$

hence, $$y^2 = 2px,$$

which is the equation of the parabola, referred to the rectangular axes, AX and AY.

Interpretation of the equation.

3. The axis of abscissas, AX, is called the *axis* of the parabola, and the origin A, is called the *vertex* of the axis, or principal *vertex.*

1. The equation of the parabola gives,

$$y = \pm \sqrt{2px}.$$

from which we see, that for every positive value of x, there will be two equal values of y, with contrary signs. Hence, *the parabola is symmetrical with respect to its axis.*

2. We see, further, that y will increase with x, and will have real values so long as x is positive. Hence, the curve extends *indefinitely*, in the direction of x positive.

If we make $x = 0$, we have,

$$y = \pm 0,$$

which shows, that the axis of Y is tangent to the curve, at the origin.

If we make x negative, we shall have,

$$y = \pm \sqrt{-2px};$$

or, y imaginary; which shows, that *the curve does not pass the axis of* Y, *and extend on the side of* x *negative.*

3. By a course of reasoning similar to that in Bk. III., (Art. **4**—5,) we have the conditions for determining the position of a point, with respect to the curve. They are,

For a point without the curve,	$y^2 - 2px > 0.$
For a point in the curve,	$y^2 - 2px = 0.$
For a point within the curve,	$y^2 - 2px < 0.$

Parameter.

4. The *Parameter* of the axis, is the double ordinate through the focus.

1. If, in the equation of the parabola,

$$y^2 = 2px,$$

we make, $x = \frac{p}{2}$, the corresponding value of y, will be the ordinate through the focus. Under this supposition, we have,

$$y^2 = 2p \times \frac{p}{2} = p^2; \quad \text{or,} \quad y = p.$$

Hence, $\quad 2p =$ the parameter.

2. In the ellipse, the parameter of the transverse axis is a third proportional to the axes (Bk. III., Art. **11**); in the parabola, it is a third proportional to any abscissa and the corresponding ordinate. For, from the equation,

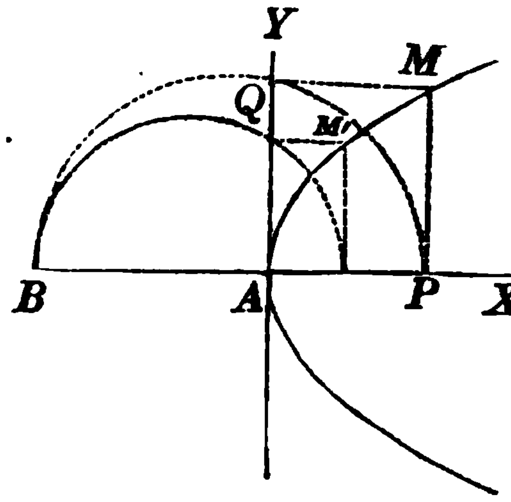
$$y^2 = 2px,$$

we have,

$$x : y :: y : 2p.$$

3. If the parameter and axis of the parabola are known,

we have a simple construction for determining points of the curve.



Let AX, AY, be the co-ordinate axes. The equation of the curve is,

$$y^2 = 2px.$$

From the origin A, lay off a distance AB, on the negative side of abscissas, equal to $2p$. Then, from A, lay off any distance, as AP, and draw PM perpendicular to AX. On BP, as a diameter, describe a semi-circumference, and through Q, where it intersects the axis AY, draw QM parallel to AX: The point M, where it intersects PM, will be a point of the curve. For, from the equation of the circle,

$$\overline{AQ}^2 = BA.AP,$$

hence, $$y^2 = 2px,$$

for any point, M or M'.

Relation of the ordinates and abscissas.

5. Denote any two ordinates of the curve, by y' and y'', and the corresponding abscissas, by x' and x''. We shall then have, from the equation of the curve,

$$y'^2 = 2px', \quad \text{and} \quad y''^2 = 2px'';$$

hence, $$y'^2 : y''^2 :: x' : x'',$$

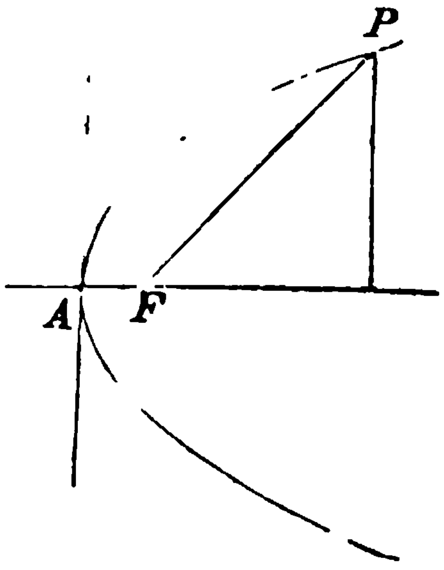
by omitting the common factor $2p$; that is,

The squares of the ordinates are to each other as their corresponding abscissas.

Polar Equation.

6. Let us resume the consideration of Equation (1) (Art. **2**), which is,

$$FP = r = \frac{p}{2} + x, \quad . \quad . \quad . \quad . \quad (1.)$$



and in which the origin of co-ordinates is at the vertex of the axis. The formulas for transferring the origin to the focus, whose co-ordinates are, $a = \frac{p}{2}$, and $b = 0$ (Bk. I., Art. **28**), are,

$$x = \frac{p}{2} + x', \quad \text{and} \quad y = y'.$$

Substituting this value of x, in Equation (1), we have,

$$r = \frac{p}{2} + \frac{p}{2} + x' = p + x' \quad . \quad . \quad . \quad (2.)$$

If we denote the variable angle which the radius-vector makes with the axis, by v, we have,

$$x' = r \cos v; \quad \text{hence,} \quad r = p + r \cos v;$$

whence,
$$r = \frac{p}{1 - \cos v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

which is the *polar equation* of the parabola, when the pole is at the focus.

1. We see, from Equation (2), that the *radius-vector is expressed, rationally, in terms of the abscissa of the point in which it intersects the curve.* This property is peculiar to the focus.

Interpretation of the polar equation.

7. In the polar equation,

$$r = \frac{p}{1 - \cos v},$$

as well as in the corresponding equation of the ellipse, which is expressed under a similar form (Bk. III., Art. 6–1), the values of the radius-vector begin at the remote vertex, that is, in the case of the parabola, at an infinite distance from the focus.

If we make $v = 0$, we have,

$$r = \frac{p}{0} = \infty.$$

If we make $v = 90°$, we have,

$$r = p.$$

that is, half the parameter.

If we make $v = 180°$, we have,

$$r = \frac{p}{2} = FA.$$

1. If it is desirable that the values of r should begin at the nearest vertex, make $v = 180° - v'$, and we shall have,

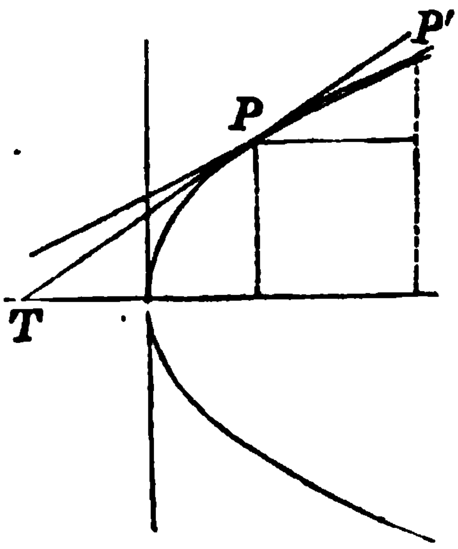
$$\cos v = -\cos v'.$$

Substituting, $-\cos v'$ for $\cos v$, the equation becomes,

$$r = \frac{p}{1 + \cos v'},$$

in which equation, the values of r begin at the nearest vertex, and v' increase from 0 to 360°.

Tangent line to the parabola.



8. Let us designate the co-ordinates of any point of the curve, as P, by x'', y''; the equation of a straight line passing through this point will be,

$$y - y'' = a(x - x'') \quad . \quad . \quad (1.)$$

It is required to determine a, when the right line is tangent to the parabola. The equation of the parabola is,

$$y^2 = 2px;$$

and, since the point of tangency is on the curve, we also have,

$$y''^2 = 2px''.$$

Subtracting the last equation from the preceding, we obtain,

$$(y + y'')(y - y'') = 2p(x - x'') \quad . \quad . \quad (2.)$$

Combining Equations (1) and (2), we have,

$$(y + y'')\,a\,(x - x'') = 2p(x - x'');$$

or, transposing and factoring,

$$(x - x'')\,[a(y + y'') - 2p] = 0;$$

an equation which may be satisfied by making,

$$x - x'' = 0, \qquad \text{or,} \qquad a(y + y'') - 2p = 0.$$

The first equation corresponds to P; in the second, x and y are the co-ordinates of P'. When P' coincides with P,

$$a = \frac{p}{y''}.$$

Substituting this value, in the equation of the line passing through P, we have,

$$y - y'' = \frac{p}{y''}(x - x'');$$

and, by reducing, and observing that $y''^2 = 2px''$,

$$yy'' = p(x + x''),$$

which is the equation of the tangent.

Sub-tangent.

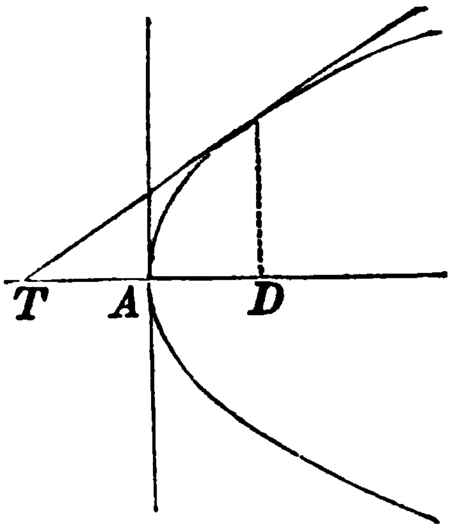
9. If, in the equation of the tangent,

$$yy'' = p(x + x''),$$

we make $y = 0$, we shall have,

$$0 = p(x + x'');$$

but since the factor p, is a constant quantity,



$$x + x'' = 0; \qquad \text{or,} \qquad x'' = -x;$$

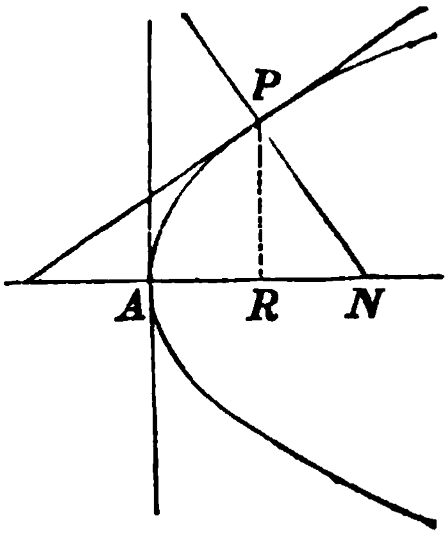
that is, AD, the abscissa of the point of tangency, is equal to $-AT$; or, the sub-tangent TD, is bisected at the vertex A.

The analytical condition, expressed by

$$x + x'' = 0, \qquad \text{or,} \qquad AT + AD = 0,$$

indicates, that the quantities are numerically equal with contrary signs; hence, they are estimated on different sides of the origin.

Normal and Sub-normal.



10. Let x'', y'', be the co-ordinates of the point of tangency. Then, the equation of the normal will be of the form,

$$y - y'' = a'(x - x''),$$

and since it is perpendicular to the tangent,

$$aa' + 1 = 0.$$

But we have already found (Art. 8),

$$a = \frac{p}{y''}; \qquad \text{hence,} \qquad a' = -\frac{y''}{p}$$

therefore, we have,

$$y - y'' = -\frac{y''}{p}(x - x''),$$

which is the equation of the normal.

1. If, in the equation of the normal,

$$y - y'' = -\frac{y''}{p}(x - x''),$$

we make $y = 0$, and then find the value of $x - x''$, we shall have,

$$x - x'' = p.$$

But, x is equal to the distance AN, and x'' to the distance AR; hence, $x - x'' = RN = p$; that is, *the sub-normal is constant, and equal to half the parameter.*

6

Perpendicular from the focus to the tangent.

11. The equation of a line passing through the focus, whose co-ordinates are, $x' = \frac{p}{2}$, and $y' = 0$, is,

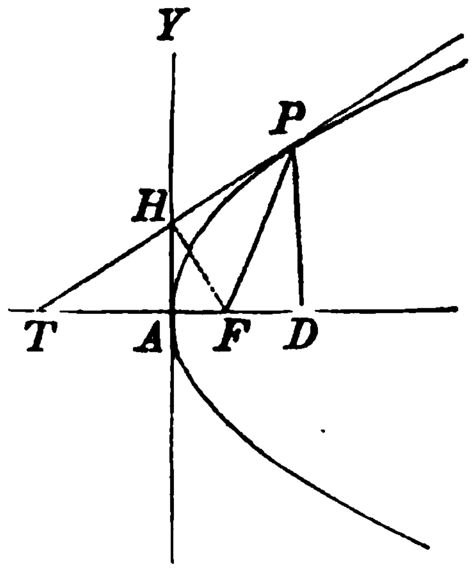
$$y = a'\left(x - \frac{p}{2}\right).$$

The condition, that this line shall be perpendicular to the tangent, gives,

$$aa' + 1 = 0;$$

hence,
$$a' = -\frac{1}{a} = -\frac{1}{\frac{p}{y''}} = -\frac{y''}{p};$$

the equation of the perpendicular HF, therefore, becomes,



$$y = -\frac{y''}{p}\left(x - \frac{p}{2}\right).$$

Combining this with the equation of the tangent TP, which is,

$$yy'' = p(x + x''),$$

and substituting for y''^2, its value $2px''$, and reducing, we find,

$$x(2x'' + p) = 0;$$

an equation which can only be satisfied when $x = 0$; hence, *the point H, at which the perpendicular meets the tangent, is on the axis of Y.*

12. Through P, the point of contact, draw PD perpendicular to the axis; then (Art. 9),

$$TA = AD; \quad \text{and hence,} \quad TH = HP.$$

Therefore, the two right-angled triangles TFH and HFP, have the two sides about the right angle equal; consequently, the triangles are equal, and the angle FTP is equal to TPF; that is,

The tangent to the parabola at any point of the curve, makes equal angles with the axis and with the line drawn from the point of tangency to the focus.

13. In the right-angled triangle TFH, in which AH is perpendicular to TF, we have,*

$$\overline{FH}^2 = FT \times FA; \quad \text{or,} \quad \overline{FH}^2 = FP \times FA.$$

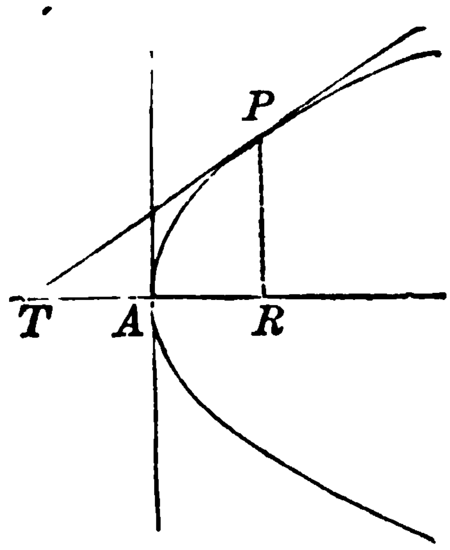
But, FA is equal to $\frac{1}{2}p$; hence, it is constant for every position of the point of contact; therefore, $\overline{FH}^2$ varies as the distance FP; that is,

The square of the perpendicular drawn from the focus to the tangent, varies, as the distance from the focus to the point of contact.

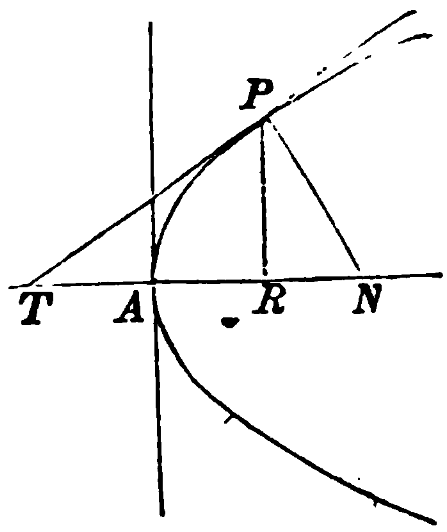
CONSTRUCTION OF TANGENT LINES.

Tangent line at a given point of the curve.

14.—1.—First Method. Let P be the given point. Draw PR perpendicular to the axis. Then, from the vertex A, lay off AT equal to AR, and join T and P; TP will be tangent to the curve (Art. 9).



* Legendre, Bk. IV. Prop. 23. Cor.

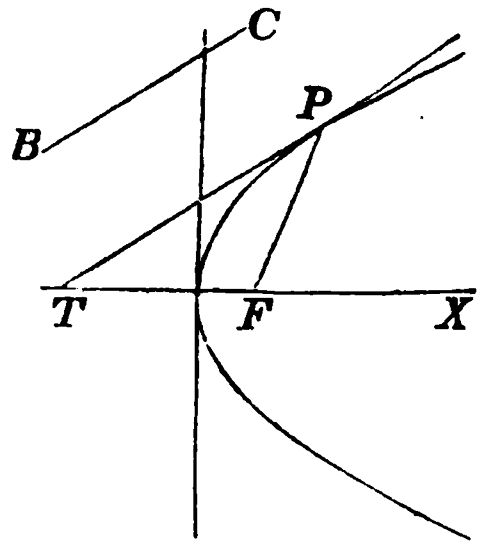


2.—**Second Method.** Draw the ordinate PR to the axis, and from the foot R, lay off a distance $RN = p$, and join P and N. Then, draw TP perpendicular to PN at P, and it will be the tangent required (Art. **10**).

3.—**Third Method.** Join P and the focus F (next figure). Then lay off from F, on the axis, a distance FT, equal to FP, and join P and T; PT will be the required tangent. (Art. **12**.)

Tangent parallel to a given line.

15. Let BC be a given line, to which a tangent is to be drawn parallel. At the focus F, lay off an angle XFP, equal to twice the angle which the given line makes with the axis of X.



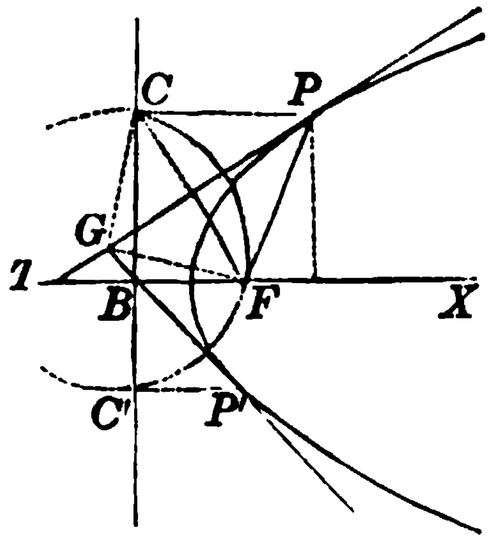
Through P, the point at which FP intersects the curve, draw PT parallel to BC, and it will be the tangent required.

For, the outward angle PFX is equal to the sum of the angles T and TPF.* But PTF being equal to the angle which BC makes with the axis of X, is equal to one-half of PFX; hence, the angle PTF is equal to half of PFX; therefore, the triangle PTF is isosceles, and, consequently, PT is tangent to the curve at P (Art. **12**).

* Legendre, Bk. I. Prop. 25. Cor. 6.

Tangent through a given point without the curve.

16. Let G be a given point, through which a tangent is to be drawn.



With G, as a centre, and a radius equal to GF, the distance to the focus, describe the arc of a circle intersecting the directrix at C and C'. Through C and C', draw two lines parallel to the axis BX, intersecting the parabola in P and P'. Through G, draw GP and GP', and they will be tangents to the parabola, at P and P'.

For, join P and the focus F. Then, since P is a point of the parabola, $PF = PC$; and, by construction, $GF = GC$; hence, the line GP has two points, G and P, each equally distant from C and F; it is, therefore, perpendicular to CF.* Since the triangle CPF is isosceles, PG bisects the angle CPF; therefore, $PTF = FTP$; hence, TP is tangent to the curve (Art. **12**).

It may be proved that GP' is tangent at P'.

PARABOLA REFERRED TO OBLIQUE AXES.

17. We have thus far deduced the properties of the parabola, from its equation, obtained by referring the curve to a system of rectangular co-ordinates, having their origin at the vertex. We now propose to develop some of the properties of the curve, by referring it to a system of oblique co-ordinates.

* Legendre, Bk. I. Prop. 26. Cor.

Equation when referred to oblique axes.

18. The formulas for passing from rectangular to oblique co-ordinates, when the origin is changed (Bk. I., Art. **29**—1), are,

$$x = a + x' \cos\alpha + y' \cos\alpha', \qquad y = b + x' \sin\alpha + y' \sin\alpha'$$

Substituting these values of x and y, in the equation,

$$y^2 = 2px,$$

it becomes,

(1.)

$$\left.\begin{array}{l} y'^2 \sin^2\alpha' + 2x'y' \sin\alpha \sin\alpha' + x'^2 \sin^2\alpha + b^2 - 2ap \\ \quad + 2(b \sin\alpha' - p \cos\alpha')y' + 2(b \sin\alpha - p \cos\alpha)x' \end{array}\right\} = 0.$$

In this equation, there are four arbitrary constants, a, b, α and α', to which we can assign values at pleasure. By giving a fixed value to either, we introduce one condition into the coefficients of Equation (1), and by assigning values to all, we introduce four conditions.

Let the values assigned to these arbitrary constants be such, that Equation (1) shall contain only the second power of y, and the first power of x; that is, reduce to the form,

$$y^2 = 2px.$$

This requires that,

$$b^2 - 2ap = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$\sin^2\alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

$$\sin\alpha \sin\alpha' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$b \sin\alpha' - p \cos\alpha' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

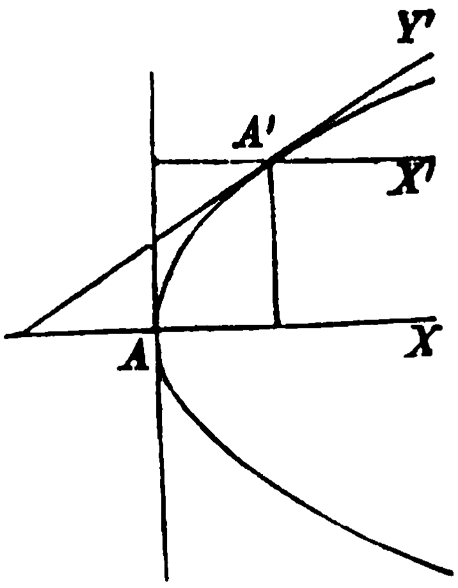
Having introduced these conditions, the equation becomes,

$$y'^2 = \frac{2p}{\sin^2\alpha'}x'.$$

Let us interpret these four conditions, separately.

Interpret the equation, $b^2 - 2ap = 0.$

1. This equation of condition is of the same form as the equation of the parabola, referred to the primitive axes. Therefore, the co-ordinates of the new origin will satisfy the primitive equation, and hence, the new origin is on the curve at some point as A'.



Interpret the equation, $\sin^2\alpha = 0.$

2. In this equation of condition, we have,

$$\sin^2\alpha = 0; \qquad \text{hence,} \qquad \alpha = 0,$$

which shows, that the new axis of abscissas, X', is parallel to the primitive axis AX.

Interpret the equation, $\sin\alpha \sin\alpha' = 0.$

3. This equation of condition, is satisfied by virtue of the $\sin\alpha = 0$; hence, it is nothing more than the second.

Interpret the equation, $b \sin\alpha' - p \cos\alpha' = 0.$

4. This equation of condition, gives,

$$\tan \alpha' = \frac{p}{b};$$

and since this value of $\tan \alpha'$ is the same as that found in (Art. 8), for the tangent of the angle which the tangent makes with the axis of X, we conclude that the new axis Y', is tangent to the parabola at the new origin, A'.

Interpret the equation, $y^2 = \frac{2p}{\sin^2\alpha'} x'.$

19. To simplify the form, put,

$$\frac{2p}{\sin^2\alpha'} = 2p';$$

we shall then have, by omitting the accents of the variables,

$$y^2 = 2p'x,$$

for the equation of the parabola, referred to the new axes. The coefficient, $2p'$, or its equal, $\frac{2p}{\sin^2\alpha'}$, is called the *parameter*, of the new diameter $A'X'$.

In this equation, every value of x will give two equal values of y, with contrary signs; hence, the curve is symmetrical with respect to the axis $A'X'$; or, *this axis bisects all chords of the parabola which are parallel to the tangent* $A'Y'$.

1. *Diameter*, as a general term, designates any straight line which bisects a system of chords drawn parallel to

the tangent at the vertex, and terminating in the curve; and the curve is said to be *symmetrical* with respect to the diameter, whether the chords are oblique or perpendicular to it. In this sense, therefore, every line drawn parallel to the axis AX, is a diameter of the parabola; hence, *all diameters of the parabola are parallel to each other, a property which shows that the centre of the curve is at an infinite distance from the vertex.*

For the area of the parabola, see Calculus, page 72.

BOOK V.

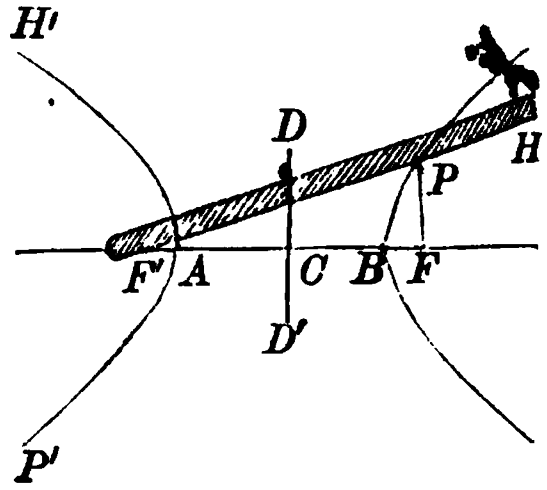
OF THE HYPERBOLA.

1. AN HYPERBOLA is a plane curve, such that the difference of the distances from any point of it to two fixed points, is equal to a given distance.

The fixed points are called the *foci*.

The characteristic property of the hyberbola, gives rise to the following constructions of the curve.

First—By a continuous movement.



2. Let F' and F, be the foci, and $F'F$, the distance between them. Take a ruler, longer than the distance $F'F$, and fasten one of its extremities at the focus F'. At the other extremity, H, attach a thread of such a length, that the length of the ruler shall exceed the length of the thread by a given distance AB. Attach the other extremity of the thread at the focus F.

Press a pencil, P, against the ruler, and keep the thread constantly tense, while the ruler is turned around F', as a centre. The point of the pencil will describe one branch of the curve.

For, $PF + PH$ is equal to the length of the thread, to

which if we add AB, we shall have the length of the ruler. Hence,

$$F'P + PH = FP + PH + AB;$$

or, $$F'P - FP = AB;$$

therefore, P is a point of the hyperbola.

If the extremity of the ruler, attached at the focus F', be removed, and attached at F, the second branch of the hyperbola, $H'AP'$, may be described in a similar manner.

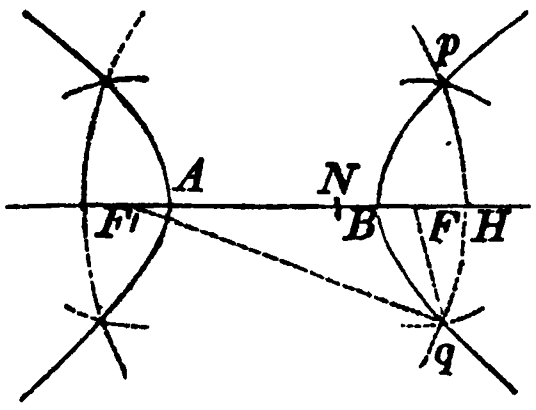
3. The TRANSVERSE AXIS is that part of the line drawn through the foci, lying between the two branches of the curve, as AB. The points A and B, in which the transverse axis intersects the curves, are called *vertices*.

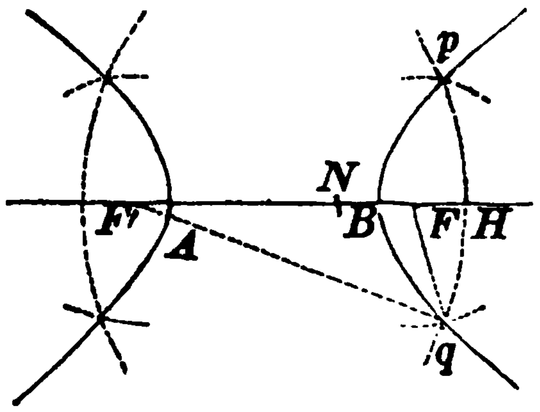
4. The CENTRE of the hyperbola, is the middle point C, of the transverse axis.

5. The line CD, drawn through the centre, perpendicular to the transverse axis, and equal to the square root of $\overline{CF}^2 - \overline{CB}^2$, is called the semi-conjugate axis; and DD', is the conjugate axis.

Second—Construct the curve by points.

6. Let AB be a given line, and F' and F, two given points. It is required to describe an hyperbola, of which AB shall be the transverse axis, and F' and F, the foci.





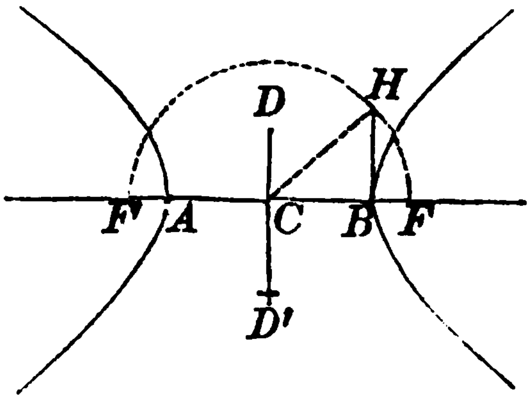
From the focus F', lay off a distance $F'N$, equal to the transverse axis, and take any other distance, as $F'H$, greater than $F'B$.

With F' as a centre, and $F'H$ as a radius, describe the arc of a circle. Then, with F as a centre, and NH as a radius, describe an arc intersecting the arc before described, at p and q; these will be points of the hyperbola; for, $F'q - Fq$, is equal to the transverse axis AB.

If, with F as a centre, and $F'H$ as a radius, an arc be described, and a second arc be described, with F' as a centre, and NH as a radius, two points in the other branch of the curve will be determined. Hence, by changing the centres, each pair of radii will determine two points in each branch.

Third—When the axes are given.

7. Since the square of the semi-conjugate axis is equal to the square of the distance from the centre to the focus minus the square of the semi-transverse axis, the square of the distance from the centre to the focus (Art. **5**), is equal to the sum of the squares of the semi-axes.



Let AB and DD', be the axes of an hyperbola.

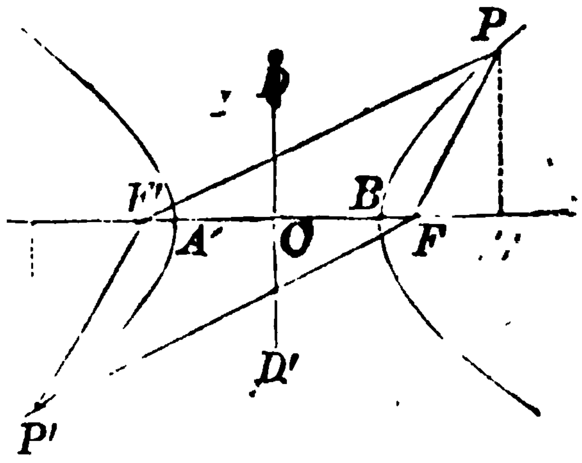
At the vertex B, draw BH perpendicular to AB, and make it equal to the semi-conjugate axis CD, or CD'. Join H and the centre C. Then, with C

as a centre, and CH as a radius, describe a semi-circumference, intersecting AB produced in F and F'; these points will be the foci.

The curves may then be described as before.

Equation of the Hyperbola.

8. Let F and F', be the foci, and denote the distance between them by $2c$. Denote the semi-transverse axis CB, by A, and the semi-conjugate, CD, by B. Let P be any point of the curve, and designate the distance $F'P$, by r', and FP, by r; then, $2A$ will denote the given line AB, to which the difference, $F'P - PF$, is to be equal.



Through the centre C, draw CD perpendicular to $F'F$, and let C be the origin of a system of rectangular co-ordinates. Let x and y denote the co-ordinates of any point, as P.

The square of the distance between any two points of which the co-ordinates are x, y, and x', y' (Bk. I., Art. **19**), is,

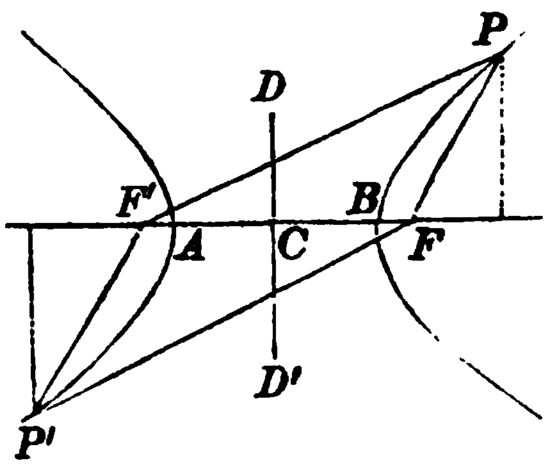
$$(y - y')^2 + (x - x')^2.$$

If the distance be estimated from the point F', of which the co-ordinates are, $x' = -c$, $y' = 0$, we shall have,

$$\overline{F'P}^2 = r'^2 = y^2 + (x + c)^2 \quad . \quad . \quad . \quad (1.)$$

and if it be estimated from the point F, of which the co-ordinates are $x' = +c$, and $y' = 0$, we shall have,

$\overline{FP}^2 = r^2 = y^2 + (x-c)^2 \ldots (2.)$



If we add and subtract Equations (1) and (2), we obtain,

$r'^2 + r^2 = 2(y^2 + x^2 + c^2) \ldots (3.)$

and, $\quad r'^2 - r^2 = 4cx \ldots (4.)$

Equation (4) may be put under the form,

$$(r' + r)(r' - r) = 4cx \quad . \quad . \quad . \quad . \quad (5.)$$

But we have, from the property of the hyperbola,

$$r' - r = 2A \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

Combining (5) and (6), we have,

$$r' + r = \frac{2cx}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

Combining (6) and (7), by addition and subtraction, we obtain,

$$r' = A + \frac{cx}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

and,
$$r = -A + \frac{cx}{A} \quad . \quad . \quad . \quad . \quad (9.)$$

Squaring both members of Equations (8) and (9), combining the resulting equations, and substituting the values of r'^2 and r^2, in Equation (3), we have,

$$A^2 + \frac{c^2x^2}{A^2} = y^2 + x^2 + c^2.$$

Substituting for c^2, its value, $A^2 + B^2$ (Art. 7), we have,

$$A^4 + A^2x^2 + B^2x^2 = A^2y^2 + A^2x^2 + A^4 + A^2B^2;$$

whence, $$A^2y^2 - B^2x^2 = -A^2B^2,$$

which is the equation of the hyperbola, referred to its centre and axes.

Interpretation of the equation.

9.—1. If in the equation of the hyperbola,

$$A^2y^2 - B^2x^2 = -A^2B^2,$$

we make $y = 0$, the corresponding values of x will be the abscissas of the points in which the curve intersects the axis of X (Bk. II., Art. 4—1); viz.:

$$x = +A, \text{ for } B; \quad \text{and,} \quad x = -A, \text{ for } A.$$

2. If we make $x = 0$, the corresponding values of y will be the ordinates of the points in which the curve intersects the axis of Y, viz.:

$$y = +B\sqrt{-1}, \text{ for } D; \quad \text{and,} \quad y = -B\sqrt{-1}, \text{ for } D';$$

and since these values are both imaginary, the curve does not intersect the conjugate axis (Introduction, p. 24).

3. If we place the equation of the hyperbola under the form,

$$y = \pm\frac{B}{A}\sqrt{x^2 - A^2}, \text{ we see,}$$

1st. That, for every value of $x < A$, whether positive

or negative, the corresponding values of y will be imaginary.

2d. That, for every value of $x > A$, whether positive or negative, there will be two values of y, numerically equal, with contrary signs.

Hence, we see, 1st. That both branches of the curve are symmetrical with respect to the axis of X. 2d. That they do not approach nearer to the centre, than the vertices B and A. 3d. That from the vertices, they extend indefinitely in the direction of x positive and x negative.

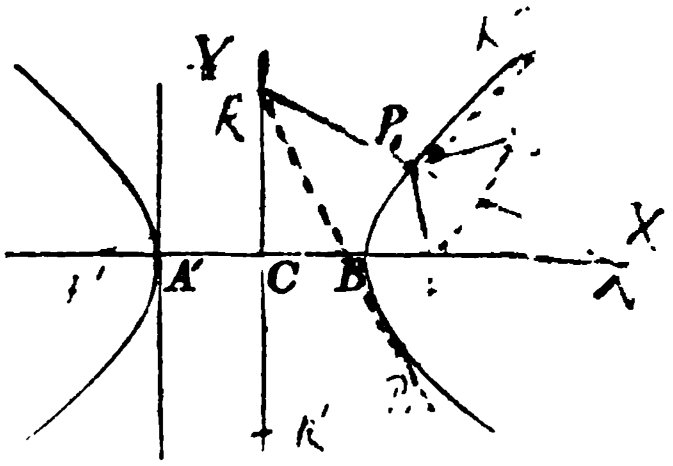
4. By a course of reasoning similar to that pursued in (Bk. III., Art. 4—5), we find the following analytical conditions for determining the position of a point with respect to the hyperbola:

Without the curve,	$A^2y^2 - B^2x^2 + A^2B^2 > 0.$
In the curve,	$A^2y^2 - B^2x^2 + A^2B^2 = 0.$
Within the curve,	$A^2y^2 - B^2x^2 + A^2B^2 < 0.$

5. By comparing the equation of the hyperbola with the equation of the ellipse, referred to its centre and axes, it is seen, that the two are *identical*, except in the sign of B^2, which is minus in the hyperbola, and plus in the ellipse. We may, therefore, pass from one equation to the other, by substituting for B, $B\sqrt{-1}$. Hence, it follows, that

Every result obtained from the equation of the ellipse, will become a corresponding result from the equation of the hyperbola, by changing B into $B\sqrt{-1}$.

Equation when the origin is at either vertex of the transverse axis.



10. If we transfer the origin of co-ordinates from the centre C to A, one extremity of the transverse axis, the equations of transformation (Bk. I., Art. **28**), will reduce to,

$$x = -A + x', \qquad y = y'.$$

Substituting these values in the equation of the hyperbola, it reduces to,

$$A^2y'^2 - B^2x'^2 + 2B^2Ax' = 0;$$

which may be put under the form,

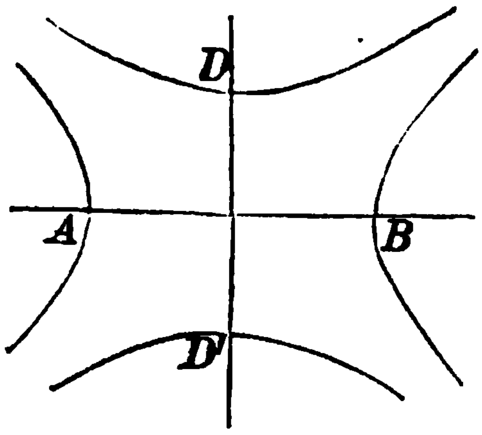
$$y'^2 = \frac{B^2}{A^2}(x'^2 - 2Ax'),$$

which is the equation of the hyperbola, referred to the vertex A, as an origin of co-ordinates.

If we refer it to the vertex B, as an origin, the equation will become,

$$y'^2 = \frac{B^2}{A^2}(2Ax' + x'^2).$$

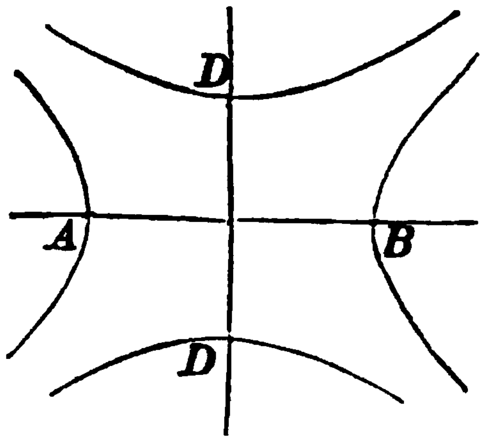
Conjugate and equilateral hyperbolas.



11. If on the conjugate axis DD', as a transverse, and a focal distance equal to $\sqrt{A^2 + B^2}$, we construct the two branches of an hyperbola, their equation will be,

$$B^2x^2 - A^2y^2 = -A^2B^2;$$

or, $$A^2y^2 - B^2x^2 = A^2B^2,$$



in which B denotes the semi-transverse axis, and A the semi-conjugate. Two hyperbolas, thus connected, are called *conjugate hyperbolas*. The conjugate axis of the one, is the transverse axis of the other, and the focal distances are equal.

1. If the transverse and conjugate axes are equal, the hyperbolas are called *Equilateral*. The equation then becomes,

$$y^2 - x^2 = -A^2, \text{ when } A \text{ is the transverse axis,}$$

and $$x^2 - y^2 = -B^2, \text{ when } B \text{ is the transverse axis.}$$

These correspond to the case in which the ellipse becomes the circle.

Eccentricity.

12. The ECCENTRICITY of the hyperbola, is the distance from the centre to either focus, divided by the semi-transverse axis, and is denoted by e; hence, $c = Ae$.

Polar Equation.

13. Resuming Equations (8) and (9), (Art. **8**), we have,

$$r' = A + \frac{cx}{A}, \qquad \text{and} \qquad r = -A + \frac{cx}{A};$$

or, $$r' = A + ex, \ldots (1.) \qquad \text{and} \qquad r = -A + ex \ldots (2.)$$

In the first of these equations, the pole lies without the curve, and in the second, within it.

1. If the origin of co-ordinates be transferred to the focus F', whose co-ordinates are, $-c = -Ae$, and 0, we have the formulas (Bk. I., Art. **28**),

$$x = -Ae + x', \qquad \text{and} \qquad y = y'.$$

Substituting this value of x, in Equation (1), and $r' \cos v$, for x', we have,

$$r' = A - Ae^2 + er' \cos v;$$

whence,
$$r' = \frac{A(1 - e^2)}{1 - e \cos v}, \quad \ldots\ldots \quad (3.)$$

which is the polar equation of a branch of the hyperbola, in terms of the eccentricity and variable angle, when the pole is *without the curve.*

2. If the origin of co-ordinates be transferred to the focus F, whose co-ordinates are, $+c = Ae$, and 0, (Bk. I., Art. **28**), we have the formulas,

$$x = Ae + x', \qquad \text{and} \qquad y = y'.$$

Substituting this value of x, in Equation (2), and $r \cos v$, for x', we have,

$$r = -A + Ae^2 + er \cos v;$$

whence,
$$r = -\frac{A(1 - e^2)}{1 - e \cos v}, \quad \ldots\ldots \quad (4.)$$

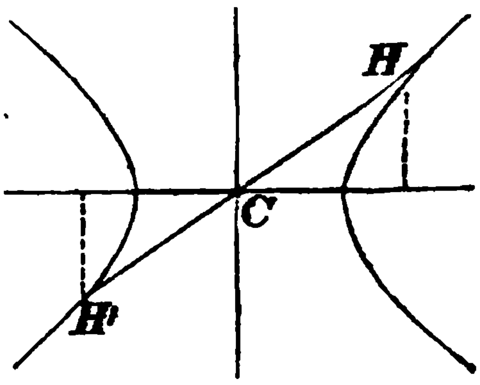
which is the polar equation of a branch of the hyperbola, in terms of the eccentricity and variable angle, when the pole is *within the curve.*

3. We see, from Equations (1) and (2), that the values of r' and r, are expressed, rationally, in terms of the abscissas of the points in which the radius-vector intersects the curve. This property is peculiar to the focus, as a pole.

Diameters.

14. A DIAMETER of any hyperbola is a line drawn through the centre and limited by the curve. The points in which it intersects the curve, are called *vertices of the diameter.*

Every diameter is bisected at the centre.



15. If, in the equations expressing the values of x', y', x'', y'' (Bk. III., Art. 8), we substitute for B, $B\sqrt{-1}$, we have,

$$x' = \quad AB\sqrt{\frac{-1}{A^2a^2 - B^2}}, \qquad y' = \quad AB\,a\sqrt{\frac{-1}{Aa^2 - B^2}}.$$

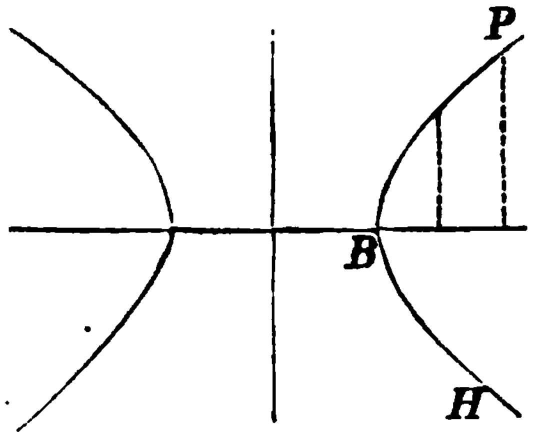
$$x'' = -AB\sqrt{\frac{-1}{Aa^2 - B^2}}, \qquad y'' = -AB\,a\sqrt{\frac{-1}{Aa^2 - B^2}}.$$

Hence, the co-ordinates of H' and H, are equal, with contrary signs, therefore, $CH = CH'$.

Relation of ordinates to each other.

16. The equation of the hyperbola, referred to the vertex B of the transverse axis (Art. 10), is,

$$y^2 = \frac{B^2}{A^2}(2Ax + x^2).$$



If we designate a particular ordinate by y', and its abscissa by x', and a second ordinate by y'', and its abscissa by x'', we shall have,

$$y'^2 = \frac{B^2}{A^2}(2Ax' + x'^2), \quad \text{and} \quad y''^2 = \frac{B^2}{A^2}(2Ax'' + x''^2).$$

Dividing the first equation by the second, we obtain,

$$\frac{y'^2}{y''^2} = \frac{(2A + x')x'}{(2A + x'')x''}; \qquad \text{or,}$$

$$y'^2 \ : \ y''^2 \ :: \ (2A + x')x' \ : \ (2A + x'')x'',$$

in which the segments, are,

$$2A + x', \ x', \quad \text{and} \quad 2A + x'', \ x''.$$

Hence, *the squares of the ordinates are as the rectangles of the segments.*

Parameter.

17. The PARAMETER of the transverse axis, is the double ordinate passing through the focus.

To find its value, let us take the Polar Equation (4) (Art. **13**),

$$r = -\frac{A(1 - e^2)}{1 - e\cos v}.$$

If we make $v = 90°$, we have (Bk. III., Art. **11**),

$$r = -A(1 - e^2).$$

Substituting for e^2, its value $\frac{A^2+B^2}{A^2}$, and reducing,

$$r = \frac{B^2}{A}; \quad \text{hence,} \quad \text{Parameter} = \frac{2B^2}{A} = \frac{4B^2}{2A}.$$

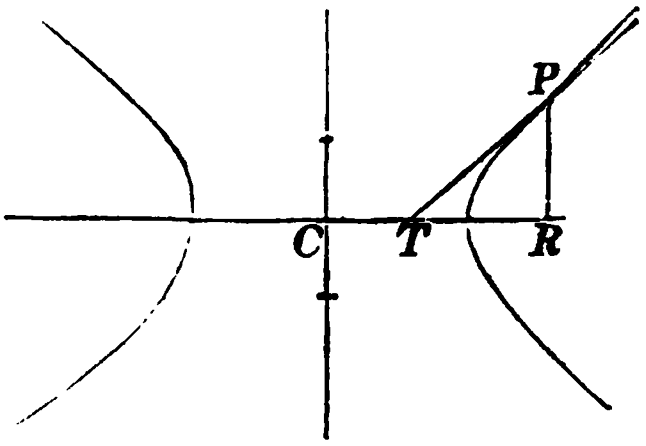
If we write this in a proportion, we have,

$$2A : 2B :: 2B : \text{Parameter}; \quad \text{that is,}$$

The parameter of the transverse axis, is a third proportional to the transverse axis and its conjugate.

1. The numerator, in the value of r' or r (Art. **13**), is equal to half the parameter.

Equation of the tangent. Sub-tangent.



18. The equation of a tangent to an ellipse, at a point whose co-ordinates are x'', y'' (Bk. III., Art. **14**), is

$$A^2yy'' + B^2xx'' = A^2B^2.$$

This will become the equation of a tangent to the hyperbola, if we substitute for B, $B\sqrt{-1}$ (Art. **9**—5); this substitution gives,

$$A^2yy'' - B^2xx'' = -A^2B^2;$$

which is the equation of a tangent to the hyperbola.

1. If, in this equation, we make $y = 0$, we have,

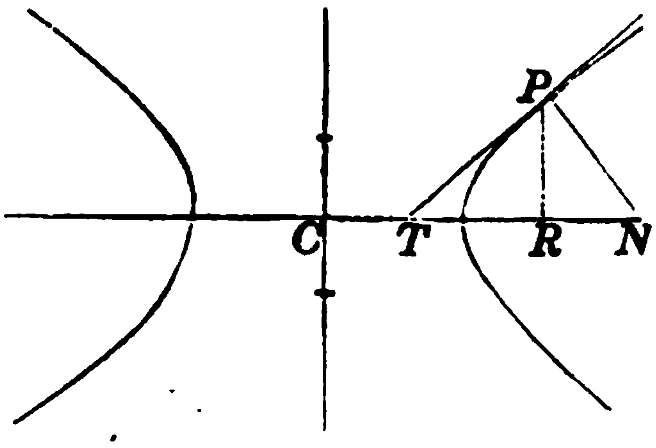
$$CT = x = \frac{A^2}{x''}.$$

Subtracting this from $CR = x''$ we obtain,

$$TR = \frac{x''^2 - A^2}{x''},$$

which is the value of the sub-tangent.

Equation of the normal. Sub-normal.



19. The equation of a normal line to an ellipse, at a point whose co-ordinates are x'', y'' (Bk. III., Art. **16**), is

$$y - y'' = \frac{A^2y''}{B^2x''}(x - x'').$$

This becomes, by changing the sign of B^2,

$$y - y'' = -\frac{A^2y''}{B^2x''}(x - x''),$$

which is the equation of the normal, PN.

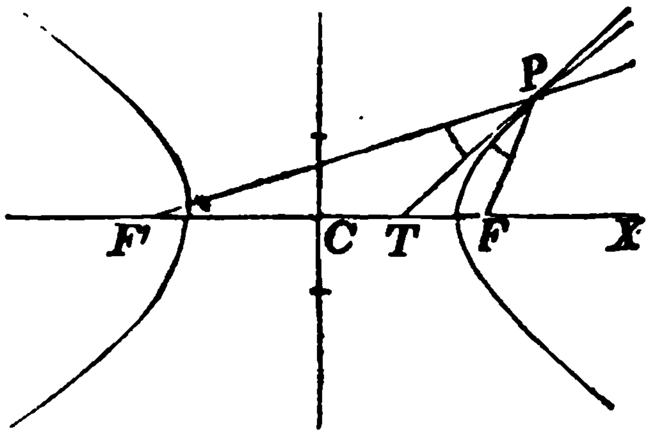
1. To find the point in which the normal intersects the axis of X, make $y = 0$, and we have,

$$CN = x = \frac{A^2 + B^2}{A^2}x'',$$

and by subtracting x'', we find the sub-normal,

$$RN = \frac{B^2x''}{A^2}.$$

Tangent bisects the angle of the two lines drawn to the foci.



20. If from P, any point of the curve, we draw two lines to the foci F' and F, and recollect that CF' or CF is equal to $c = Ae$ (Art. **13**), we have, by using the value of $CT = \frac{A^2}{x''}$ (Art. **18**),

$$F'T = Ae + \frac{A^2}{x''} = \frac{Aex'' + A^2}{x''} = \frac{A(ex'' + A)}{x''};$$

and
$$TF = Ae - \frac{A^2}{x''} = \frac{Aex'' - A^2}{x''} = \frac{A(ex'' - A)}{x''};$$

hence,
$$F'T \; : \; TF \; :: \; ex'' + A \; : \; ex'' - A.$$

By referring to the values of r' and r (Art. **8**), and remembering that $\frac{c}{A} = e$, we have,

$$r' \; : \; r \; :: \; ex'' + A \; : \; ex'' - A;$$

hence,*
$$F'T \; : \; TF \; :: \; r' \; : \; r.$$

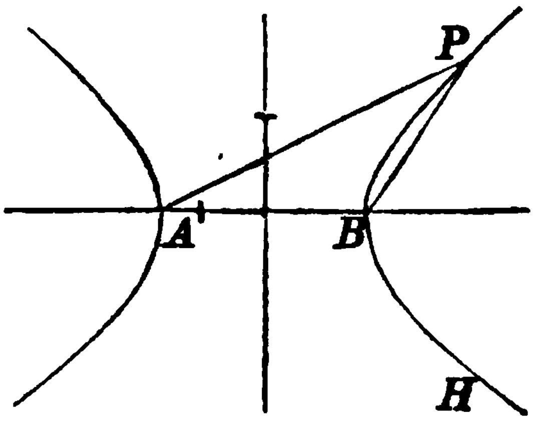
Therefore, PT bisects the angle $F'PF$.† Hence,

If a line be drawn tangent to an hyperbola at any point, and two lines be drawn from the same point to the foci, the lines drawn to the foci will make equal angles with the tangent.

* Leg., Bk. II. Prop. 4. Cor. † Leg., Bk. IV. Prop. 17.

Supplementary Chords.

21. The equation of condition of supplementary chords, in the ellipse (Bk. III., Art. **20**), is,



$$aa' = -\frac{B^2}{A^2}.$$

Substituting for B, $B\sqrt{-1}$, we have,

$$aa' = \frac{B^2}{A^2}.$$

1. If the chords are drawn to any point P, in the branch HBP, the tangents a and a' will be both positive; if drawn to a point in the other branch, they will both be negative.

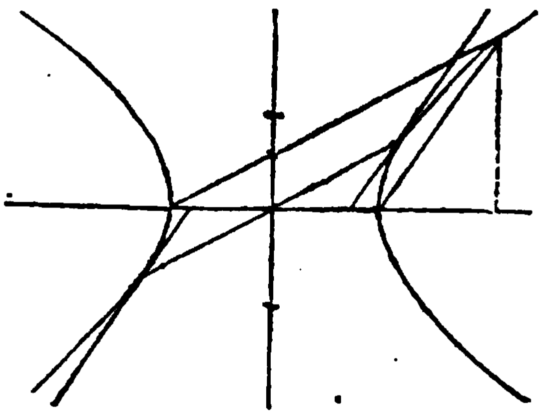
2. If tne hyperbola is equilateral, $A = B$, and there will result,

$$aa' = 1,$$

which shows, that *the sum of the two acute angles formed by the supplementary chords with the transverse axis, on the same side, is equal to* 90°.*

Supplementary Chords. Tangent and diameter.

22. In the ellipse, the relation between the tangents of the angles which a tangent and the diameter passing through the point of contact make with the transverse axis, is expressed by the equation



* Legendre, Trig. Art. 13.

(Bk. III., Art. 21),

$$aa' = -\frac{B^2}{A^2};$$

hence, in the hyperbola, it is,

$$aa' = \frac{B^2}{A^2}.$$

But the equation of condition of supplementary chords, is

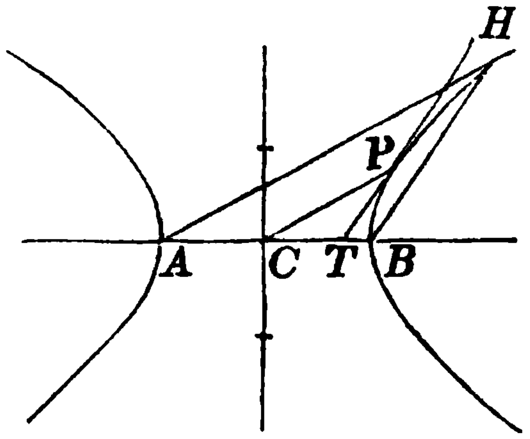
$$aa' = \frac{B^2}{A^2};$$

hence, $$aa' = aa';$$ that is,

If one chord is parallel to a diameter, the other will be parallel to the tangents drawn through its vertices.

Construction of tangent lines to the hyperbola.

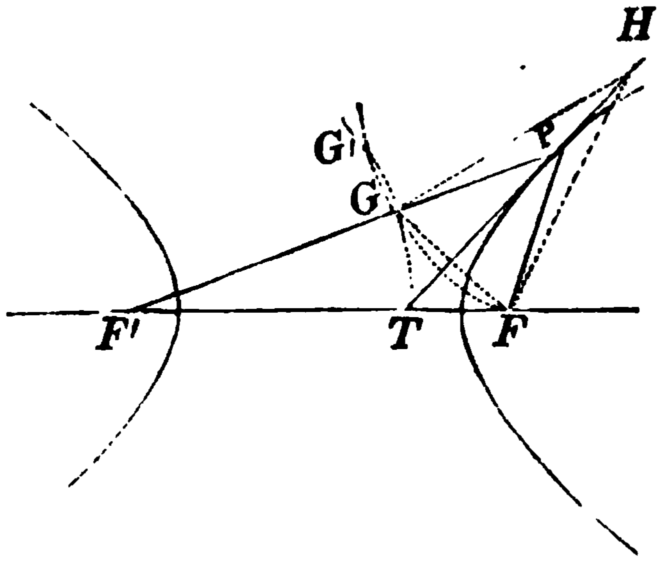
23. The property proved in Art. **22**, enables us to draw a tangent to an hyperbola, at a given point of the curve.



Let C be the centre of the hyperbola, AB its transverse axis, and P the given point of the curve at which the tangent is to be drawn.

Through P, draw the semi-diameter PC, and through A, draw the supplementary chord AH, parallel to it. Then, draw the other supplementary chord BH, and through P, draw PT parallel to BH; then will PT be the tangent required.

24. The property proved in Art. **20**, enables us to draw a tangent to an hyperbola, through a given point without the curve.



Let H be the given point. With this point as a centre, and HF, as a radius, describe the arc of a circle. With F', as a centre, and a radius equal to the transverse axis, describe the arc of a circle intersecting the former at G and G'. Draw $F'G$, cutting the curve in P. Through P, draw HPT, and it will be tangent to the hyperbola at P.

For, if we draw HF, HG, we shall have $HF = HG$, by construction; and since P is a point of the hyperbola, and $F'G$ equal to the transverse axis, we shall have $PF = PG$; hence, PT is perpendicular to FG;* and since the triangle FGP is isosceles, PT will bisect the angle $F'PF$, and will, therefore, be tangent to the hyperbola.

1. The two arcs described with the centres F' and H, intersect each other in two points, G and G'; a line may, therefore, be drawn through F' and either of these points, thus giving two points of tangency, and two tangents.

Conjugate diameters.

25. Two diameters of an hyperbola are said to be conjugate to each other, when either of them is parallel to the two tangents drawn through the vertices of the other.

* Legendre, Bk. I. Prop. 16. Cor.

26. The equation of condition of conjugate diameters in the ellipse (Bk. III., Art. **27**), is,

$$A^2 \sin\alpha \sin\alpha' + B^2 \cos\alpha \cos\alpha' = 0.$$

Hence, for the hyperbola, it is,

$$A^2 \sin\alpha \sin\alpha' - B^2 \cos\alpha \cos\alpha' = 0,$$

or, $$A^2 \tan\alpha \tan\alpha' - B^2 = 0.$$

Hyperbola referred to its centre and conjugate diameters.

27. The equation of the hyperbola, referred to its centre and axes, is

$$A^2y^2 - B^2x^2 = -A^2B^2.$$

The formulas for passing from rectangular to oblique co-ordinates, the origin remaining the same, are,

$$x = x'\cos\alpha + y'\cos\alpha', \qquad y = x'\sin\alpha + y'\sin\alpha'.$$

Squaring these values of x and y, and substituting in the equation of the hyperbola, we have,

$$\left.\begin{array}{l}(A^2\sin^2\alpha' - B^2\cos^2\alpha')y'^2 + (A^2\sin^2\alpha - B^2\cos^2\alpha)x'^2 \\ \quad + 2(A^2\sin\alpha\sin\alpha' - B^2\cos\alpha\cos\alpha')x'y'\end{array}\right\} = -A^2B^2.$$

But the condition, that the new axes shall be conjugate diameters, gives,

$$A^2 \sin\alpha \sin\alpha' - B^2 \cos\alpha \cos\alpha' = 0;$$

hence, the equation reduces to

$$(A^2\sin^2\alpha' - B^2\cos^2\alpha')y'^2 + (A^2\sin^2\alpha - B^2\cos^2\alpha)x'^2 = -A^2B^2.$$

If we suppose, in succession, $y' = 0$, $x' = 0$, and denote

by A' and B', the corresponding abscissas and ordinates, we find,

$$A'^2 = \frac{-A^2B^2}{A^2\sin^2\alpha - B^2\cos^2\alpha}, \qquad B'^2 = \frac{-A^2B^2}{A^2\sin^2\alpha' - B^2\cos^2\alpha'}.$$

If A'^2 is positive, A' will be real, and the diameter will intersect the curve. Under this supposition, we shall have,

$$A^2\sin^2\alpha < B^2\cos^2\alpha, \qquad \text{or,} \qquad \tan\alpha < \frac{B}{A}.$$

But, $$\tan\alpha\tan\alpha' = \frac{B^2}{A^2} \quad (\text{Art. } \mathbf{26});$$

hence, $$\tan\alpha' > \frac{B}{A}; \qquad \text{or,} \qquad A^2\sin^2\alpha' > B^2\cos^2\alpha';$$

hence, B'^2 will be negative.

The supposition, therefore, which renders A'^2 positive, or A' real, gives B'^2 negative, or B' imaginary; which shows that only one of the diameters intersects the curve. Attributing to B'^2, its proper sign, we have,

$$A'^2 = \frac{-A^2B^2}{A^2\sin^2\alpha - B^2\cos^2\alpha}, \qquad -B'^2 = \frac{-A^2B^2}{A^2\sin^2\alpha' - B^2\cos^2\alpha'}.$$

Finding the values of the denominators in these equations, substituting them in the general equation, and reducing, we obtain,

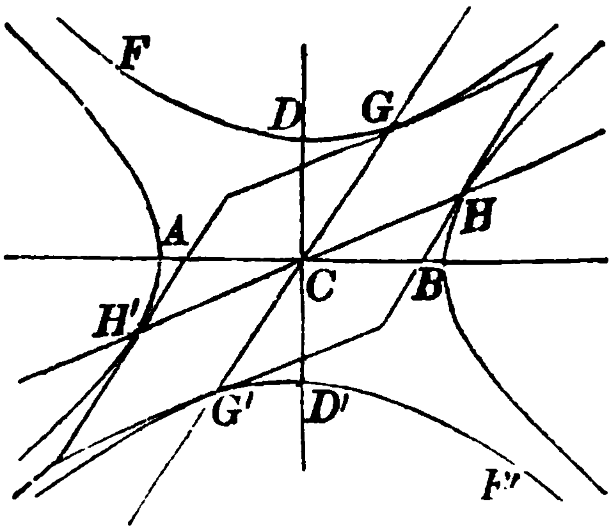
$$A'^2y'^2 - B'^2x'^2 = -A'^2B'^2;$$

or, omitting the accents of x and y, since they are general variables, we obtain,

$$A'^2y^2 - B'^2x^2 = -A'^2B'^2,$$

which is the equation of the hyperbola, referred to its centre and conjugate diameters.

Interpretation.



1. We have already seen (Art. 9—2), that when the transverse axis AB is real, the conjugate axis DD' will be imaginary, and reciprocally; that is, the two axes will not intersect the same branch of the hyperbola. The last proposition proves the same property for any two conjugate diameters.

If then, $2A'$ designates the diameter $H'H$, $2B'$ will designate the conjugate diameter $G'G$, terminating in the conjugate hyperbola; and each will be parallel to the two tangent lines drawn through the vertices of the other.

If B' were made real, A' would be imaginary, and the equation would represent the curves FDG, $F'D'G'$.

2. The equation of the hyperbola, referred to its centre and conjugate diameters, being of the same form as when referred to its centre and axes, it follows, that every value of x, will give two equal values of y, with contrary signs; or, if B' were real, every value of y, would give two equal values of x, with contrary signs; hence, each hyperbola is symmetrical with respect to the diameter which it intersects; that is,

Either diameter bisects all chords drawn parallel to the other, and terminated by the curve.

3. It may be readily shown, that the squares of the ordinates to either diameter, are proportional to the rect-

angles of the corresponding segments, from the foot of the ordinates respectively, to the vertices of the diameter.

4. The equations of the hyperbola and ellipse, referred to their centres and conjugate diameters, are *identical*, except in the sign of B'^2, which is minus in the hyperbola, and plus in the ellipse. We may, therefore, pass from one equation to the other, by substituting for B', $B'\sqrt{-1}$. Hence, it follows, that, *every result obtained from the equation of the ellipse, will become a corresponding result from the equation of the hyperbola, by changing* B' *into* $B'\sqrt{-1}$.

Relation between the axes and conjugate diameters.

28. In the ellipse (Bk. III., Art. **31**), we have,

$$A'B' \sin(\alpha' - \alpha) = AB; \text{ and,}$$

$$A'^2 + B'^2 = A^2 + B^2.$$

By substituting for B', $B'\sqrt{-1}$, and for B, $B\sqrt{-1}$, we have, for the hyperbola,

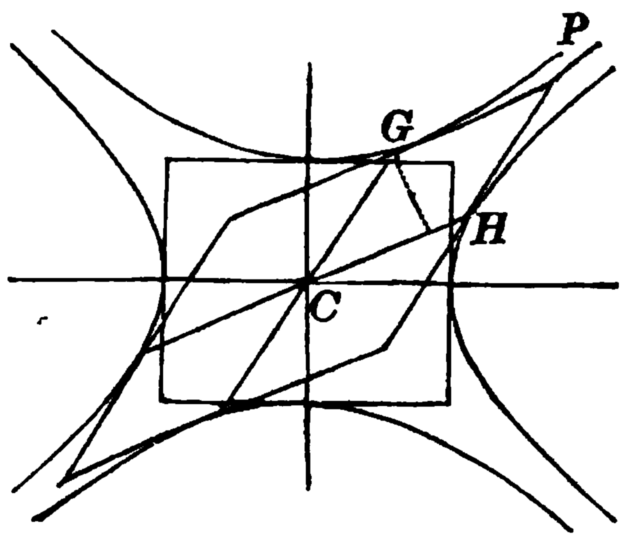
$$A'B' \sin(\alpha' - \alpha) = AB \quad . \quad . \quad . \quad . \quad (1.)$$

and,

$$A'^2 - B'^2 = A^2 - B^2 \quad . \quad . \quad (2.)$$

Interpretation.

Equation (1).—Construct a rectangle on the axes, and a parallelogram on two conjugate diameters. Draw from G a perpendicular to CH; this perpendicular will be equal to $B' \sin(\alpha' - \alpha)$. Hence, the area of the paral-



lelogram $CGPH$, is equal to $A'B' \sin(\alpha' - \alpha) = AB$; hence, the whole parallelogram is equal to the whole rectangle. Therefore, *the parallelogram formed by drawing tangents at the four vertices of conjugate diameters, is equivalent to the rectangle formed by drawing tangents through the vertices of the axes.*

2. Equation (2).—This equation,

$$A'^2 - B'^2 = A^2 - B^2,$$

or,

$$4A'^2 - 4B'^2 = 4A^2 - 4B^2,$$

expresses this property: *The difference of the squares of two conjugate diameters is equal to the difference of the squares of the axes.*

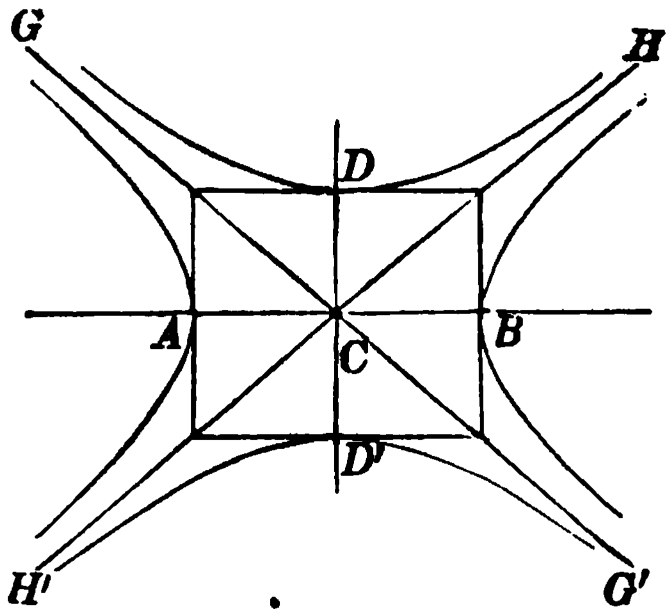
Hence, there can be no equal conjugate diameters, unless $A = B$; in which case, the hyperbola is equilateral, and then, *every diameter will be equal to its conjugate.*

The hyperbola referred to its asymptotes.

29. The *Asymptotes* of an hyperbola, are the diagonals of the rectangle described on the axes, indefinitely produced in both directions.

Thus, $H'H$, $G'G$, are asymptotes of the hyperbola whose transverse axis is AB, and also of the conjugate

hyperbola whose transverse axis is DD'.



If we designate the angle estimated from CB around to CH, by α, and the angle BCG, by α'; or, what is equivalent, designate the angle BCG', by $-\alpha'$,

$$\tan\alpha = \frac{B}{A} \quad . \quad . \quad . \quad (1.) \qquad \tan\alpha' = -\frac{B}{A} \quad . \quad . \quad . \quad (2.)$$

or,

$$\tan^2\alpha = \frac{B^2}{A^2}, \qquad \tan^2\alpha' = \frac{B^2}{A^2}; \text{ or,}$$

$$A^2\sin^2\alpha - B^2\cos^2\alpha = 0. \quad (3.) \qquad A^2\sin^2\alpha' - B^2\cos^2\alpha' = 0. \quad (4.)$$

Equations (3) and (4) express the relations between the angles which the asymptotes form with the transverse axis. They are called, *equations of condition.*

1. If the hyperbola is equilateral, $A = B$, and Equations (1) and (2), give,

$$\tan\alpha = 1, \qquad \text{and} \qquad \tan\alpha' = -1;$$

which shows, that *the asymptotes make equal angles with the transverse axis—one lying in the first and third angles, and the other in the second and fourth; and that they are at right angles to each other.*

2. Since the sine of the angle at the base is equal to the perpendicular divided by the hypothenuse; and the cosine, to the base divided by the hypothenuse,* we have,

* Legendre, Trigonometry, Art. 32.

$$\sin\alpha = \frac{B}{\sqrt{A^2+B^2}}. \quad (1.) \qquad \sin\alpha' = \frac{-B}{\sqrt{A^2+B^2}}. \quad (2.)$$

$$\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}. \quad (3.) \qquad \cos\alpha' = \frac{A}{\sqrt{A^2+B^2}}. \quad (4.)$$

Equation of the curve referred to its asymptotes.

30. The equation of the hyperbola, referred to its centre and axes is,

$$A^2y^2 - B^2x^2 = -A^2B^2.$$

The formulas for passing from rectangular to oblique co-ordinates, the origin remaining the same, are,

$$x = x'\cos\alpha + y'\cos\alpha', \qquad y = x'\sin\alpha + y'\sin\alpha'.$$

Substituting these values, and reducing, we obtain,

$$\left.\begin{array}{l}(A^2\sin\alpha' - B^2\cos^2\alpha')y'^2 + (A^2\sin^2\alpha - B^2\cos^2\alpha)x'^2 \\ \quad + 2(A^2\sin\alpha\sin\alpha' - B^2\cos\alpha\cos\alpha')x'y'\end{array}\right\} = -A^2B^2.$$

The equations of condition (Art. **29**), reduce the coefficients of x'^2, and y'^2, to 0. Multiplying Equations (1) and (2) (Art. **29**—2), and (3) and (4), and reducing, the coefficient of $x'y'$, becomes,

$$-\frac{4A^2B^2}{A^2+B^2};$$

hence, the equation of the hyperbola, referred to its centre and asymptotes, is,

$$x'y' = \frac{A^2+B^2}{4},$$

or, by putting M, for $\dfrac{A'^2 + B'^2}{4}$, and omitting the accents,

$$xy = M.$$

Interpretation of the equation.

31. If lines be drawn through the vertices of the axes, they will form the rhombus $AD'BD$. The diagonals CP, CQ, of the rectangles described on the semi-axes, are equal to each other, and each is equal to $\sqrt{A^2 + B^2}$. But these are also equal to the diagonals BD, BD', and each pair mutually bisect each other at H and N.

Hence, $CH = \frac{1}{2}\sqrt{A^2 + B^2}$, and $CN = \frac{1}{2}\sqrt{A^2 + B^2}$;

therefore, $$CH \times CN = \frac{A^2 + B^2}{4} = xy.$$

If we designate the angle included between the asymptotes by β, we shall have,

$$CH \times CN \sin\beta = xy \sin\beta;$$

the first member of the equation is equal to the rhombus $CHBN$; the second, to any parallelogram, as $CQMK$, whose sides are denoted by x and y; that is,

The rhombus described on the abscissa and ordinate of the vertex of the hyperbola, is equivalent to the parallelo-

gram described on the abscissa and ordinate of any point of the curve.

1. The rhombus $CHBN$, described on the abscissa and ordinate of the vertex of the hyperbola, is called the *power of the hyperbola.* It is one-eighth of the rectangle described on the axes.

Asymptotes approach the curve.

32. Let us resume the equation of Art. **30**,

$$xy = M, \quad \text{from which,} \quad y = \frac{M}{x}.$$

Since M is constant, if x increases continually, y will diminish, and if x becomes infinite, y will become 0; hence, the hyperbola continually approaches the asymptote, and as y cannot become negative so long as x is positive, it follows that the curve will touch the asymptote when y is 0. The same may be shown with respect to the other asymptote. Hence,

The asymptotes continually approach tne hyperbola, and become tangent to it at an infinite distance from the centre.

Asymptote, the limit of tangents.

33. The equation of the tangent, when the curve is is referred to its centre and axes (Art. **18**), is,

$$A^2yy'' - B^2xx'' = -A^2B^2.$$

If we make $y = 0$, we find,

$$x = \frac{A^2}{x''},$$

which is the distance from the centre to the point in which the tangent intersects the transverse axis.

If x'' increases, x diminishes, and if x'' be made infinite, x will be equal to 0; that is, the tangent line will pass through the centre, and since both the tangent and asymptote touch the curve at a point infinitely distant from the centre, they will coincide.

1. The asymptotes have been defined as the diagonals, prolonged, of the rectangle described on the axes. It is easily proved, that they are also the *common diagonals of all parallelograms formed by drawing tangent lines through the vertices of conjugate diameters.*

ALGEBRAIC CURVES.

Classification.

34. An ALGEBRAIC CURVE is one in which the relation between the co-ordinates of all its points are expressed only in algebraic terms.

35. We have seen that every equation of the first degree, between two variables, is the equation of a straight line (Bk. I., Art. **18**).

We have also seen, that the equations of the circle, the ellipse, the parabola, and the hyperbola, are all of the second degree; and analogy would lead us to infer, that *every equation of the second degree between two variables,* represents one or the other of these curves. This is now to be proved rigorously.

36. The general equation of the second degree between two variables, is,

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \quad . \quad (1.)$$

which contains the first and second powers of each variable, their product, and an absolute term, F.

The coefficients, A, B, C, D, E, and F, are entirely independent of the variables y and x. They are called *constants;* or *arbitrary* constants, since values may be attributed to them at pleasure.

37. Let us suppose that the co-ordinate axes are rectangular; this supposition will not render the discussion, or the results, less general. For, if the co-ordinate axes were oblique, we might readily pass to a system of rectangular co-ordinates, without affecting the *degree of the equation*, since the equations for transformation are always linear.

Change of direction of the axes.

38. To pass from a system of rectangular to a system of oblique co-ordinates, the origin remaining the same, we have (Bk. I., Art. **29**),

$$x = x'\cos\alpha + y'\cos\alpha', \qquad y = x'\sin\alpha + y'\sin\alpha'.$$

Substituting these values of x and y, in Equation (1), we have,

$$(2.)$$

$$\left.\begin{array}{r} A\sin^2\alpha' \\ B\sin\alpha'\cos\alpha' \\ C\cos^2\alpha' \\ \end{array}\right| y'^2 + \left.\begin{array}{r} 2A\sin\alpha\sin\alpha' \\ B\sin\alpha\cos\alpha' \\ B\sin\alpha'\cos\alpha \\ 2C\cos\alpha\cos\alpha' \end{array}\right| x'y' + \left.\begin{array}{r} A\sin^2\alpha \\ B\sin\alpha\cos\alpha \\ C\cos^2\alpha \\ \end{array}\right| x'^2$$

$$+ \left.\begin{array}{r} D\sin\alpha' \\ E\cos\alpha' \end{array}\right| y' + \left.\begin{array}{r} D\sin\alpha \\ E\cos\alpha \end{array}\right| x' + F = 0$$

Since α and α' are entirely arbitrary, we may assign to either of them, such a value as will reduce the co-efficient of $x'y'$ to 0. This supposition gives,

$$2A\sin\alpha\sin\alpha' + B(\sin\alpha\cos\alpha' + \sin\alpha'\cos\alpha) + 2C\cos\alpha\cos\alpha' = 0.$$

If we suppose, $\alpha' - \alpha = 90°$, the new axes will be rectangular, and we shall have,

$$\sin\alpha' = \cos\alpha, \qquad \text{and} \qquad \cos\alpha' = -\sin\alpha.$$

Making the substitutions, we have,

$$2A\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha) - 2C\sin\alpha\cos\alpha = 0;$$

or, $$(A - C)2\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha) = 0.$$

But,

$$2\sin\alpha\cos\alpha = \sin 2\alpha, \qquad \text{and} \qquad \cos^2\alpha - \sin^2\alpha = \cos 2\alpha;*$$

hence, $$(A - C)\sin 2\alpha + B\cos 2\alpha = 0.$$

Dividing both members of the equation, by $\cos 2\alpha$, we have,

$$\tan 2\alpha = -\frac{B}{A - C}.$$

Therefore, when the new axis X', makes with the primitive axis X, an angle equal to half the angle whose tangent is $-\frac{B}{A - C}$, the coefficient of $x'y'$ will reduce to 0.

Equation (2) will then take the form, omitting the accents,

$$A'y^2 + C'x^2 + D'y + E'x + F = 0 \quad . \quad . \quad (4.)$$

* Legendre, Trig. Art. 34.

Change of the origin of co-ordinates.

39. The formulas for passing from a system of co-ordinates to a parallel system (Bk. I., Art. **28**), are,

$$x = a + x', \qquad \text{and} \qquad y = b + y'.$$

Substituting these values of x and y, in Equation (4), we have,

$$A'y'^2 + C'x'^2 + \begin{array}{r|} 2A'b \\ D' \end{array}\; y' + \begin{array}{r|} 2C'a \\ E' \end{array}\; x'$$

$$+ A'b^2 + C'a^2 + D'b + E'a + F = 0 \quad . \quad . \quad (5.)$$

In Equation (5), a and b are entirely arbitrary. If we attribute to them such values as make,

$$2A'b + D' = 0, \qquad \text{whence,} \qquad b = -\frac{D'}{2A'}, \quad . \quad . \quad (6.)$$

and, $2C'a + E' = 0$, whence, $a = -\dfrac{E'}{2C'}$, . . (7.)

and put, $-(A'b^2 + C'a^2 + D'b + E'a + F) = F'$, . . (8.)

Equation (5) will become, omitting the accents,

$$A'y^2 + C'x^2 = F' \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Interpretation.

40.—1. The transformation, from Equation (1) to (4), is always possible. For, such a value may be given to α or α', as shall render the coefficient of $x'y'$, in Equation (2), equal to 0.

2. The transformation, from Equation (4) to (8), is

always possible, except in the cases when b and a, in Equations (6) and (7), are both infinite, or when either of them is infinite. Under the first supposition, $A' = 0$, and $C' = 0$, which causes the second powers of the variables to disappear, in Equation (4), and the equation then becomes,

$$D'y + E'x + F = 0,$$

which is the equation of a straight line (Bk. I., Art. **18**).

If only one of them is infinite, as a, for example, then, $C' = 0$, and Equation (5), after making $2C'a + E' = Q$, takes the form,

$$A'y^2 + Qx = F''.$$

If we now transfer the origin of co-ordinates to a point on the axis of X, such that,

$$x = \frac{F''}{Q} - x',$$

we shall have, $A'y^2 + Q\left(\frac{F''}{Q} - x'\right) = F''$;

or, $y^2 = \frac{Q}{A'}x'$; or, $y^2 = 2px$,

which is the equation of a parabola.

Every curve, denoted by an equation of the form,

$$y^n = 2px,$$

in which n is any positive number, except 1, is called a parabola.

If $n = 2$, we have the common parabola. If $n = 3$, the *cubic parabola*, &c.

3. Let us interpret Equation (8),

$$A'y^2 + C'x^2 = F'.$$

When A', C', and F', are all positive, this is the equation of an ellipse, referred to its centre and axes (Bk. III., Art. 3); then, $A' = A^2$, $C' = B^2$, and $F' = A^2B^2$. If $A' = C'$, it becomes the equation of the circumference of a circle.

4. If A' is negative, and C' and F' positive, then, by changing the signs of both members,

$$A'y^2 - C'x^2 = -F',$$

which is the equation of an hyperbola referred to its centre and axes; then, $A' = A^2$, $C' = B^2$, and $F' = A^2B^2$ (Bk. IV., Art. 8).

5. If A' is negative, C' positive, and F' negative, then, by changing the signs, we have,

$$A'y^2 - C'x^2 = F'.$$

which is the equation of a conjugate hyperbola; $C' = B^2$, $A' = A^2$, $F' = A^2B^2$, and $2B$ the transverse axis (Bk. V., Art. 11).

6. If, in Equation (2), we attribute such values to α and α', as shall reduce the coefficients of the second powers of the variables to 0; and then transfer the origin of co-ordinates, so as to get rid of the first powers of the variables, the equation will take the form,

$$x'y' = M,$$

which is the equation of an hyperbola, referred to its centre and asymptotes (Bk. V., Art. 30). Hence,

Every equation of the second degree between two variables, will, under every hypothesis, represent either a circle, an ellipse, a parabola, or an hyperbola.

41.—1. Lines are classed into orders, according to the degree of their equations.

2. Straight lines are represented by equations of the first degree, between two variables, and are called, *lines of the first order.*

3. The circumference of the circle, the ellipse, the parabola, and the hyperbola, are represented by equations of the second degree, between two variables; hence, they are called, *lines of the second order.*

4. And lines denoted by an equation of the third degree, are lines of the third order; and similarly, for the higher degrees.

Equations when the origin is in the curve.

42.—1. The equation of the circle, when the origin is in the curve, is,

$$y^2 = 2Rx - x^2.$$

2. The equation of the ellipse, when the origin is at the vertex of the transverse axis, is,

$$y^2 = \frac{B^2}{A^2}(2Ax - x^2).$$

3. The equation of the parabola, under the same hypothesis, is,

$$y^2 = 2px.$$

4. The equation of the hyperbola is,

$$y^2 = \frac{B^2}{A^2}(2Ax + x^2).$$

These equations may all be put under the form,

$$y^2 = mx + nx^2,$$

in which m is the parameter of the curve, and n the square of the ratio of the semi-axes. In the circle and ellipse, n is negative; in the hyperbola it is positive, and in the parabola it is 0.

2. The curves, whose properties have been discussed in the four last books, are precisely those which are obtained by intersecting the surface of a cone by planes, as is shown in Bk. VII., Art. 45—50. For this reason they are called, *Conic Sections.*

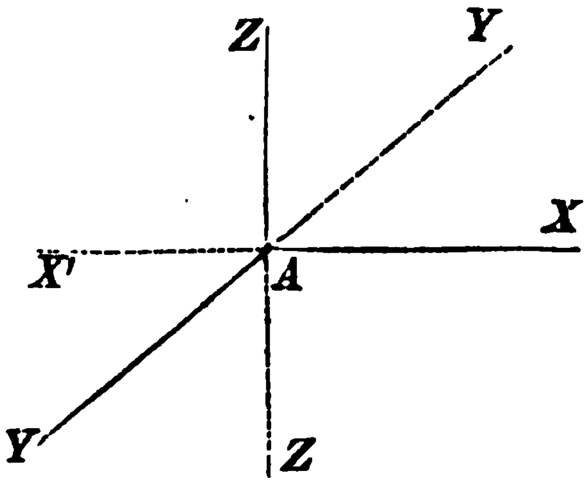
BOOK VI.

SPACE—POINT AND LINE—PLANE—SURFACES.

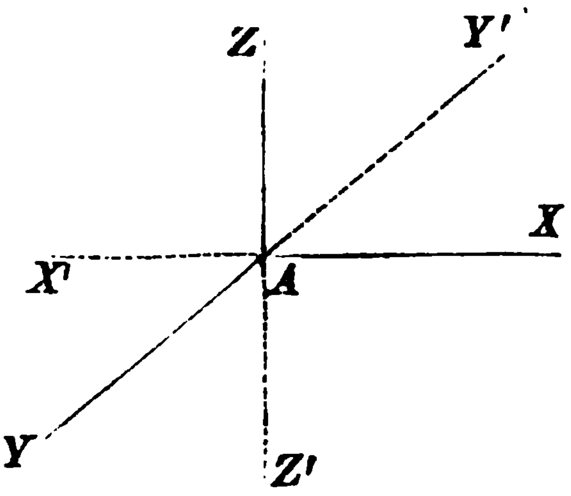
1. SPACE is indefinite extension, and is entirely similar in all its parts. The geometrical magnitudes are portions of space. Their *absolute* places cannot be determined, either by construction or by the algebraic analysis, since there is nothing fixed to which they can be referred. Their *relative* positions may, however, be easily found, and these enable us to discuss and develop their properties.

2. Thus far, the analysis has been limited to points and lines lying in the same plane. These have been referred to two straight lines, making a given angle with each other. The analysis is now to be extended to points and lines in space, which will be referred to three planes, at right angles to each other.

3. Through any point, as A, conceive a horizontal plane to be drawn. Through the same point, conceive a vertical plane, ZAX, to be drawn: this is the plane of the paper and intersects the horizontal plane in the line $X'AX$. Through the same point conceive a second ver-



tical plane to be drawn, perpendicular to the plane ZAX. This plane will intersect the horizontal plane in the line YAY', and the first vertical, in the line ZAZ'. These three planes are called, *co-ordinate planes*



4. Since the co-ordinate planes are respectively at right angles to each other, the line of intersection of either two will be perpendicular to the third: and this line of intersection is called the *axis* of that plane to which it is perpendicular.

For example: Z is the axis of the horizontal plane YX, X, the axis of the first vertical plane YAZ, and Y the axis of the second vertical plane ZAX. The three are called, the *co-ordinate axes*, and their point of intersection A, the *origin of co-ordinates.*

5. The co-ordinate planes are supposed to be indefinite, and hence, they will divide all space into eight equal parts, or triedral angles, having the origin A, for a common vertex. Four of these angles are above the horizontal plane YAX, and four below it. They are thus designated.

YAX	is called the	1st	angle.
YAX'	"	2d	"
$X'AY'$	"	3d	"
$Y'AX$	"	4th	"

The fifth angle is directly beneath the first, the sixth beneath the second, the seventh beneath the third, and the eighth beneath the fourth.

This manner of naming the angles differs from that adopted in the plane, where the first angle is beyond the axis of abscissas, and where we pass round from the right to the left; both methods are now too well established to be changed, merely for the purpose of producing uniformity.

6. The distance of any point, in space, from either of the co-ordinate planes, is estimated on the axis of that plane, or on a line parallel to the axis.

7. If from any point, in space, a line be drawn perpendicular to either of the co-ordinate planes, the foot of the perpendicular is called *the projection* of the point on that plane.

8. The line in which any plane intersects either of the co-ordinate planes, is called its *trace* on that plane.

9. If, through a straight line, in space, a plane be passed perpendicular to either of the co-ordinate planes, its trace is called, the *projection* of the line on that plane.

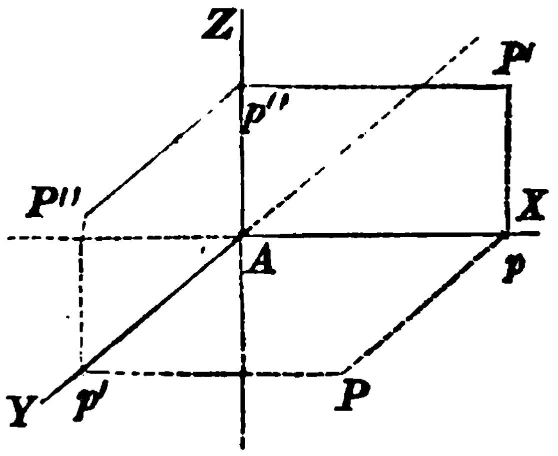
10. Let us suppose that we know the distance of a point from the three co-ordinate planes, viz.:

$$\text{from } YZ = a,$$

$$\text{from } ZX = b,$$

$$\text{from } YX = c,$$

From the origin A, lay off on the axis of X, a distance $Ap = a$, and through p pass a plane parallel to the co-ordinate plane YZ. Its traces pP', pP, will be respectively parallel to the axes Z and Y. Lay off, in like manner, on the axis of Y, a distance $Ap' = b$, and through p' pass a plane parallel to the co-ordinate plane ZX. Its traces $p'P$, $p'P''$ will be parallel, respectively, to the axes X and Z. Since the point must be in both planes, at the same time, it will be in their common intersection, which is perpendicular to the horizontal plane at P.



Lay off, from the origin of co-ordinates, on the axis of Z, a distance $Ap'' = c$, and through p'', pass a plane parallel to YX: its traces $p''P'$, $p''P''$, will be parallel, respectively, to the axes X and Y, and the point in which this plane is pierced by the perpendicular to the horizontal plane at P, will be the position of the required point. The point will, therefore, be vertically projected on the plane ZX, at P', and on the plane ZY, at P''. Its co-ordinates, are Pp', pP, and pP'.

The distances of a point, from the co-ordinate planes, are expressed, algebraically, by

$$x = a, \qquad y = b, \qquad z = c,$$

and since these conditions determine the position of the point, they are called, the *equations of the point.*

11. Let us now consider these conditions, in a general manner, and see, what each, taken separately, implies.

The conditions, $x = \pm a$,

limit the point to one of two planes drawn parallel to the co-ordinate plane YZ, on different sides of the origin, and at a distance from it equal to a.

The conditions, $y = \pm b$,

limit the point to one of two planes drawn parallel to the co-ordinate plane ZX, on different sides of the origin, and at a distance from it equal to b.

If these conditions exist together, the point will be limited to four straight lines, parallel to the axis of Z.

The conditions, $z = \pm c$,

limit the position of the point to one of two planes drawn parallel to the co-ordinate plane YX, on different sides of the origin, and at a distance from it equal to c.

If all the conditions exist at the same time, the point will be found at either one of the eight points in which the two last planes are pierced by the four parallels before determined; and each of these eight points will be found in one of the eight angles, formed by the co-ordinate planes. By attributing to the co-ordinates of these points the signs plus and minus, the position of either one of them may be exactly determined. The following signs are attributed to the co-ordinates of a point in the different angles:

1st	angle,	$x = +a.$	$y = +b,$	$z = +c,$
2d	angle,	$x = -a,$	$y = +b,$	$z = +c,$
3d	angle,	$x = -a,$	$y = -b,$	$z = +c,$
4th	angle,	$x = +a,$	$y = -b,$	$z = +c,$

5th	angle,	$x = +a,$	$y = +b,$	$z = -c,$
6th	angle,	$x = -a,$	$y = +b,$	$z = -c,$
7th	angle,	$x = -a,$	$y = -b,$	$z = -c,$
8th	angle,	$x = +a,$	$y = -b,$	$z = -c.$

12. Since either co-ordinate denotes the distance of a point from a co-ordinate plane, it follows, that when this distance is 0, the point will be found in the plane.

Hence, we have the following for the equations of the co-ordinate planes:

For the co-ordinate plane YAX, whose axis is Z,

$$z = 0, \qquad x \text{ and } y \text{ indeterminate;}$$

that is, x and y must be indeterminate, in order that they may represent the co-ordinates of every point of the plane.

For the co-ordinate plane XAZ, whose axis is Y,

$$y = 0, \qquad x \text{ and } z \text{ indeterminate.}$$

For the co-ordinate plane YAZ, whose axis is X,

$$x = 0, \qquad y \text{ and } z \text{ indeterminate.}$$

13. Since either axis lies in two of the co-ordinate planes, we shall have, for the equations of the axis of X,

$$y = 0, \quad z = 0, \quad \text{and} \quad x \text{ indeterminate.}$$

For the equations of the axis of Y,

$$x = 0, \quad z = 0, \quad \text{and} \quad y \text{ indeterminate.}$$

For the equations of the axis of Z,

$$x = 0, \quad y = 0, \quad \text{and} \quad z \text{ indeterminate.}$$

And for the origin, which lies in the three axes,

$$x = 0, \quad y = 0, \quad \text{and} \quad z = 0.$$

14. We also have, for a point in the axis of X,

$$y = 0, \quad z = 0, \quad \text{and} \quad x = \pm a.$$

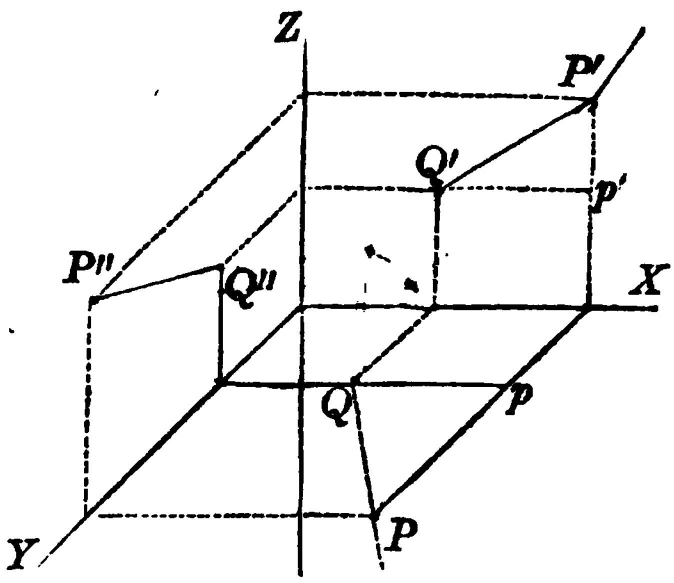
For a point in the axis of Y,

$$x = 0, \quad z = 0, \quad \text{and} \quad y = \pm b.$$

For a point in the axis of Z,

$$x = 0, \quad y = 0, \quad \text{and} \quad z = \pm c.$$

Distance between two points.



15. Let (Q, Q', Q'',) be one of the points, and (P, P', P'',) the other.

Denote the co-ordinates of the first point, by x', y', z', those of the second by x'', y'', z'', and the length of the required distance, by D. The line QP, is the projection of the given line on the co-ordinate plane of YX, $Q'P'$ its projection on ZX, and $P''Q''$ its projection on YZ. The distance D, will be the hypothenuse of a triangle, of which the base is QP, and altitude $p'P'$.

But, $Qp = x'' - x'$, $Pp = y'' - y'$, and $p'P' = z'' - z'$.

In the right-angled triangle QPp, we have,

$$\overline{QP}^2 = (x'' - x')^2 + (y'' - y')^2;$$

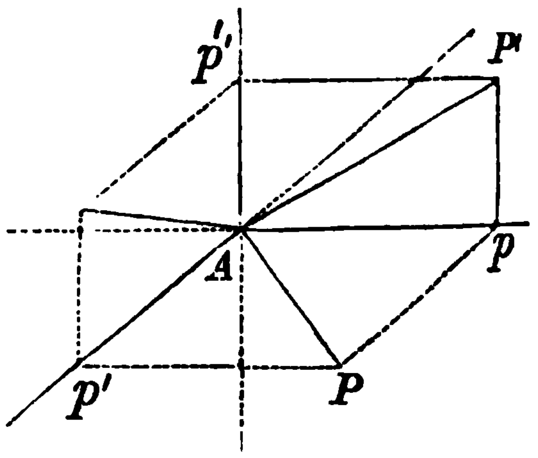
hence, $$D^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2,$$

and, $$D = \sqrt{(x'' - x')^2 + (x'' - x')^2 + (z'' - z')^2}.$$

1. The *projection* of a line, on either of the co-ordinate *axes*, is that part of the axis intercepted between the two perpendiculars drawn through its extremities. Hence, if the line whose length is D, be projected on the three co-ordinate axes, $x'' - x'$, $y'' - y'$, $z'' - z'$, will represent, respectively, the length of the projection on each axis; hence, it follows, that *the square of any line in space, is equal to the sum of the squares of its three projections on the co-ordinate axes.*

2. If one of the points, the one, for example, of which the co-ordinates are x', y', z', be placed at the origin, we shall have,

$$D = \sqrt{x''^2 + y''^2 + z''^2},$$



which expresses the distance from the origin of co-ordinates to any point in space.

Line and co-ordinate axes.

16. The three lines Pp, Pp', $P'p$, drawn perpendicular to the co-ordinate planes, may be regarded as the three

edges of a parallelopipedon, of which the line drawn to the origin is the diagonal. We have, therefore, verified a proposition of geometry, viz.: *the sum of the squares of the three edges of a rectangular parallelopipedon is equal to the square of its diagonal.*

1. This last result offers an easy method of determining a relation that exists between the cosines of the angles which a straight line makes with the co-ordinate axes.

Let us designate the length of the line, passing through the origin of co-ordinates by r, and the angles which it forms with the axes, respectively, by X, Y, and Z.

We shall then have for the lines Ap, Ap', Ap'', which are respectively designated by x'', y'', z'', the following values, viz.:

$$x'' = r\cos X, \qquad y'' = r\cos Y, \qquad z'' = r\cos Z.$$

By squaring these equations, and adding, we obtain,

$$x''^2 + y''^2 + z''^2 = r^2(\cos^2 X + \cos^2 Y + \cos^2 Z).$$

But we have already found,

$$x''^2 + y''^2 + z''^2 = r^2.$$

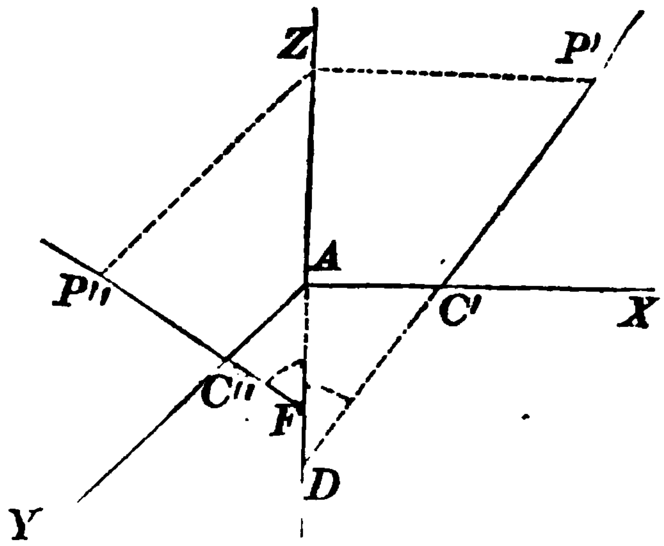
Hence, $\qquad \cos^2 X + \cos^2 Y + \cos^2 Z = 1;$

that is, *the sum of the squares of the cosines of the three angles which a straight line forms with the three co-ordinate axes, is equal to radius square, or* 1.

Equation of a straight line in space.

17. Let $C'P'$ be the projection of a straight line on

the co-ordinate plane ZX, and $C''P''$, its projection on the co-ordinate plane YZ.



Since $C'P'$ is the projection of the line on the co-ordinate plane ZX, the line itself, in space, is in the plane passing through $C'P'$, and perpendicular to the co-ordinate plane, ZX (Art. 9). Since $C''P''$ is the projection of the line on the co-ordinate plane ZY, the line itself, in space, is in the plane passing through $C''P''$ and perpendicular to the co-ordinate plane YZ ; hence, it must be the common intersection of these two projecting planes. The conditions, therefore, which fix the projections of a line, will determine the line in space.

Let $$x = az + \alpha,$$

be the equation of the projection $C'P'$, and

$$y = bz + \beta,$$

the equation of the projection $C''P''$.

In these equations, a denotes the tangent of the angle ADP', α the distance AC'; b the tangent of the angle $P''FZ$, and β the distance AC''. The angles in the co-ordinate plane ZX, are estimated from the axis Z to the right, and in the co-ordinate plane YZ, they are estimated from the axis Z, towards the left.

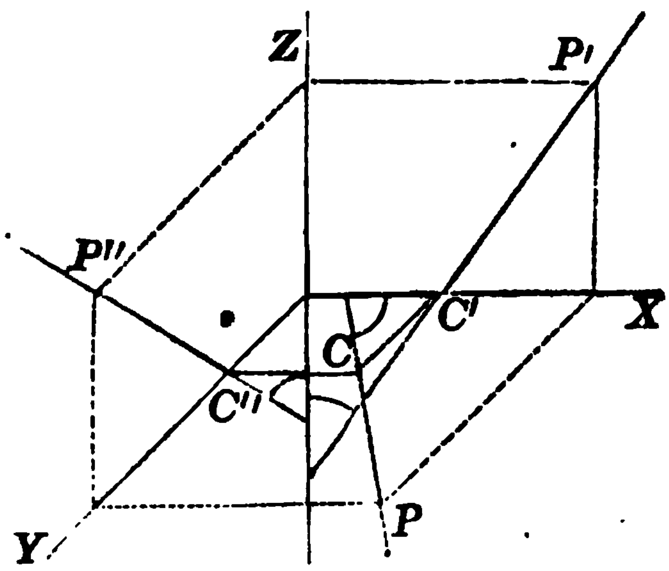
If we suppose a, α, b, and β, to be given, the two projections $C'P'$, $C''P''$, will be determined; and hence, the

line, of which they are the projections, will be determined in space. Hence,

$$x = az + \alpha, \qquad y = bz + \beta,$$

are the equations of a straight line.

1. Since the projections of a straight line on the two co-ordinate planes ZX, ZY, determine the position of the line in space, they ought, also, to determine its projection on the third co-ordinate plane, YX. This indeed may be easily proved.



For, through P' draw a parallel to the axis of Z, and from the point in which it intersects the axis of X, draw a parallel to axis of Y. Through P'' draw a parallel to the axis of Z, and through the point in which it intersects the axis of Y, draw a parallel to the axis of X; then will P be the projection of the point (P', P''), on the co-ordinate plane YX.

Find, in a similar manner, the projection of a second point, as (C', C''), and draw the projection CP.

Interpretation of the equations of a line.

18.—1. Let us now consider the equations,

$$x = az + \alpha, \qquad y = bz + \beta,$$

separately.

The equation,

$$x = az + \alpha,$$

being independent of y, will be satisfied for every point of the plane passing through $C'P'$, and perpendicular to the co-ordinate plane ZX; hence, it may be regarded as the *equation* of that plane.

In like manner, the equation,

$$y = bz + \beta,$$

being independent of x, will be satisfied for every point in the plane passing through $C''P''$, and perpendicular to the co-ordinate plane YZ; hence, it may be regarded as the *equation* of that plane.

2. Let us now consider the conditions which would be imposed upon the straight line, by supposing a, b, α, and β, to become known, in succession.

When a, b, α, and β, are all undetermined, the equations,

$$x = az + \alpha, \qquad y = bz + \beta,$$

may be made to represent every straight line which can be drawn in space, by attributing suitable values to a, b, α, and β. And when a, b, α, and β, have given values, the equations will designate but a single straight line.

If we suppose a to be given, the line may have any position in space, such, that its projection on the co-ordinate plane ZX, shall make an angle with Z, of which the tangent is a.

If we suppose α also to be given, the projection of the line on the co-ordinate plane ZX, will intersect the axis of X at a given point, and the two conditions, will limit

the line to a given plane. Its position in the plane will still be entirely undetermined.

If we now suppose b to be given, the *direction* of the line will then be determined, but it may still have an indefinite number of parallel positions in the given plane.

If finally, we attribute a value to β, the projection on the plane of YZ, will intersect the axis of Y at a given point; and hence, the position of the line will become known. The letters α and β represent the co-ordinates of the points in which the line intersects the co-ordinate plane YX.

The resolution of problems involving the straight line in space, consists in finding such values for the arbitrary constants a, b, α, and β, as shall satisfy the required conditions.

Equations of a line passing through two points.

19. Let x', y', z', and x'', y'', z'', be the co-ordinates of two given points.

The required equations will be of the form,

$$x = az + \alpha \quad . \; . \; (1.) \qquad\qquad y = bz + \beta \quad . \; . \; (2.)$$

in which it is required to find such values for a, α, b, and β, as shall cause the right line to fulfill the required conditions.

Since the straight line is to pass through a point, of which the co-ordinates are x', y', z', we shall have,

$$x' = az' + \alpha \quad . \; . \; (3.) \qquad\qquad y' = bz' + \beta \quad . \; . \; (4.)$$

and since it is also to pass through a point, of which the co-ordinates are x'', y'', z'', we shall likewise have,

$$x'' = az'' + \alpha \quad . \quad . \quad (5.) \qquad\qquad y'' = bz'' + \beta \quad . \quad . \quad (6.)$$

The last four equations enable us to determine the four constants, a, α, b, β.

By subtracting Equation (5) from Equation (3), and (6) from (4), we obtain,

$$x' - x'' = a(z' - z''), \qquad \text{and} \qquad y' - y'' = b(z' - z''),$$

from which we find,

$$a = \frac{x' - x''}{z' - z''}, \qquad \text{and} \qquad b = \frac{y' - y''}{z' - z''};$$

hence, a and b are determined. If these known values be substituted, respectively, in Equations (3) and (4), or in (5) and (6), the values of α and β will become known, and either set would represent the required line.

But it is more convenient to have the equations under another form. Subtract Equation (3) from Equation (1), and (4) from (2); we then have,

$$x - x' = a(z - z'), \qquad y - y' = b(z - z'),$$

which are the equations of a straight line passing through a given point. Substituting for a and b, their known values, we have,

$$x - x' = \frac{x' - x''}{z' - z''}(z - z'), \qquad y - y' = \frac{y' - y''}{z' - z''}(z - z'),$$

which are the equations of a straight line passing through the two given points.

Lines intersecting and parallel.

20. It is required to find the conditions which will cause two lines to intersect each other.

Let $\quad x = az + \alpha, \qquad y = bz + \beta,$

and $\quad x = a'z + \alpha', \qquad y = b'z + \beta',$

be the equations of the lines, in which the arbitrary constants a, α, b, β, a', α', b', β', are undetermined.

If these lines intersect each other in space, they must have one point in common, and the co-ordinates of this point will satisfy the equations of both lines. If we designate the co-ordinates of the common point by x', y', z', we shall have,

$$x' = az' + \alpha \quad . \; . \; (1.) \qquad y' = bz' + \beta \quad . \; . \; (2.)$$

$$x' = a'z' + \alpha' \quad . \; . \; (3.) \qquad y' = b'z' + \beta' \quad . \; . \; (4.)$$

Eliminating x' and y' from these equations, we find,

$$(a - a')z' + \alpha - \alpha' = 0 \quad . \; . \; . \; . \; (5.)$$

$$(b - b')z' + \beta - \beta' = 0 \quad . \; . \; . \; . \; (6.)$$

and if z' be eliminated from the two last equations, we have,

$$(a - a')(\beta - \beta') - (\alpha - \alpha')(b - b') = 0,$$

which is called the *equation of condition*, since it must always be satisfied in order that the two straight lines may intersect each other.

There are eight arbitrary constants entering into this

equation. It may, therefore, be satisfied in an infinite number of ways. Indeed, if values be attributed, at pleasure, to seven of the constants, such a value may, in general, be found for the remaining one, as will satisfy the equation, and, consequently, cause the lines to intersect each other.

When parallel.

21. Let us now find the co-ordinates of the point of intersection. We find, from Equations (5) and (6),

$$z' = \frac{\alpha' - \alpha}{a - a'}, \qquad \text{or} \qquad z' = \frac{\beta' - \beta}{b - b'}.$$

Substituting this value of z', in Equations (1) and (2), we have,

$$x' = \frac{a\alpha' - a'\alpha}{a - a'}, \qquad y' = \frac{b\beta' - b'\beta}{b - b'}.$$

These values of the co-ordinates of the point of intersection, become infinite, when

$$a = a', \qquad \text{and} \qquad b = b';$$

that is, when *the projections of the lines on the co-ordinate planes ZX and ZY, are parallel.*

1. If we have, at the same time,

$$\alpha' = \alpha, \qquad \text{and} \qquad \beta' = \beta,$$

the co-ordinates of the point of intersection will become $\frac{0}{0}$, or indeterminate; as, indeed, they should do, since the two lines would then coincide throughout their whole extent.

Angle between two lines.

22. Let

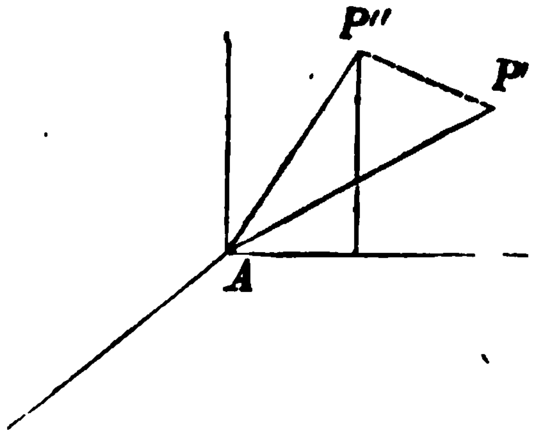
$$x = az + \alpha, \qquad y = bz + \beta,$$

be the equations of the first line, and

$$x = a'z + \alpha', \qquad y = b'z + \beta',$$

be the equations of the second.

It has been shown,* that two straight lines which cross each other in space, may be regarded as forming an angle, although they do not lie in the same plane. They are supposed to make the same angle with each other as would be formed by one of the lines, and a line drawn through any point of it, and parallel to the other; or, as would be formed by two lines drawn through the same point, and respectively parallel to the given lines.



If, then, two lines be drawn through the origin of co-ordinates, respectively parallel to the given lines, the angle which they form with each other will be equal to the required angle.

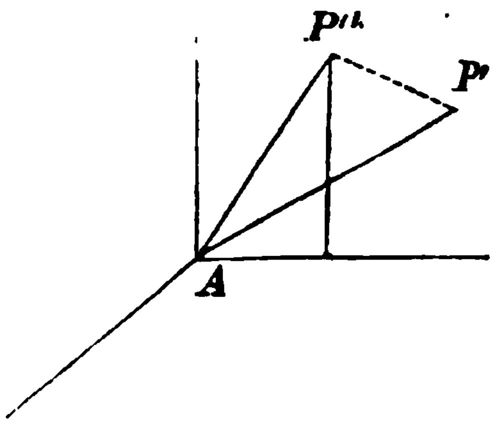
The equations of these lines will be,

$$x = az, \qquad y = bz, \qquad \text{for the first,}$$

$$x = a'z, \qquad y = b'z, \qquad \text{for the second.}$$

Let us take, on the first line, any point, as P', and designate its co-ordinates by x', y', z', and its distance from the origin, by r'. Take, in like manner, on the second line, any

* Legendre, Bk. VI. Prop. 6. Sch. 2.



point, as P'', and designate its co-ordinates by x'', y'', z'', and its distance from the origin, by r'', and let D denote the distance between the points. If we designate the angle included between the lines, by V, we shall have, in the triangle $AP'P''$,*

$$\cos V = \frac{r'^2 + r''^2 - D^2}{2r'r''}, \quad . \quad . \quad . \quad (1.)$$

and we have now only to find r', r'', and D.

Let us designate the three angles which the first line forms with the co-ordinate axes, respectively, by X, Y, and Z, and the angles which the second line forms with the same axes, by X', Y', and Z'; we shall then have (Art. 16),

$$x' = r' \cos X, \qquad y' = r' \cos Y, \qquad z' = r' \cos Z,$$

$$x'' = r'' \cos X', \qquad y'' = r'' \cos Y', \qquad z'' = r'' \cos Z'.$$

But the square of the distance between two points is,

$$D^2 = (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2 \quad \text{(Art. 15), or,}$$

$$D^2 = x'^2 + y'^2 + z'^2 + x''^2 + y''^2 + z''^2 - 2(x'x'' + y'y'' + z'z'');$$

or, by substituting for the co-ordinates of the points, their distances from the origin into the cosines of the angles which the lines make with the co-ordinate axes, we have,

$$D^2 = \left\{ \begin{array}{c} r'^2(\cos^2 X + \cos^2 Y + \cos^2 Z) + r''^2(\cos^2 X' + \cos^2 Y' + \cos^2 Z') \\ - 2r'r''(\cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z') \end{array} \right\}$$

* Leg., Trig. Art. 47.

But it has been shown (Art. 16—1), that,

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1, \qquad \cos^2 X' + \cos^2 Y' + \cos^2 Z' = 1;$$

and hence,

$$D^2 = r'^2 + r''^2 - 2r'r''(\cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z').$$

If this value of D^2 be substituted in Equation (1),

$$\cos V = \frac{r'^2 + r''^2 - D^2}{2r'r''},$$

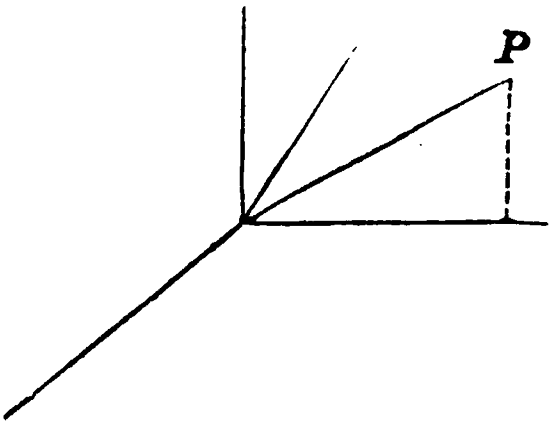
we shall find, after dividing by $2r'r''$,

$$\cos V = \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z';$$

that is, *the cosine of the angle included between two lines, is equal to the sum of the rectangles of the cosines of the angles which the lines in space form with the co-ordinate axes.*

Angle under another form.

23. Having found the cosine of the angle included between two lines, in terms of the angles which they form with the co-ordinate axes in space, we shall, in the next place, find the same value in terms of the angles which the projections of the lines on the co-ordinate planes ZX and YZ, form with the axis of Z.



The equations of the parallel lines drawn through the origin, are,

$$x = az, \qquad y = bz,$$
$$y = a'z, \qquad y = b'z.$$

Let us designate the co-ordi-

nates of the point P, on the first line, by x', y', z'; we shall then have,

$$x' = az', \qquad y' = bz';$$

and for the value of r',

$$r'^2 = x'^2 + y'^2 + z'^2.$$

From these three equations we find,

$$x' = \frac{ar'}{\sqrt{1+a^2+b^2}}, \quad y' = \frac{br'}{\sqrt{1+a^2+b^2}}, \quad z' = \frac{r'}{\sqrt{1+a^2+b^2}}$$

But we have already found (Art. 16),

$$x' = r' \cos X, \qquad y' = r' \cos Y, \qquad z' = r' \cos Z.$$

Substituting these values, and dividing by r', we obtain,

$$\cos X = \frac{a}{\sqrt{1+a^2+b^2}}, \quad \cos Y = \frac{b}{\sqrt{1+a^2+b^2}}, \quad \cos Z = \frac{1}{\sqrt{1+a^2+b^2}}.$$

If we reason, in the same manner, on the equations of the second line, we shall find,

$$\cos X' = \frac{a'}{\sqrt{1+a'^2+b'^2}}, \quad \cos Y' = \frac{b'}{\sqrt{1+a'^2+b'^2}}, \quad \cos Z' = \frac{1}{\sqrt{1+a'^2+b'^2}}.$$

If these values be now substituted in equation for $\cos V$, we shall have,

$$\cos V = \frac{1 + aa' + bb'}{\pm\sqrt{1 + a^2 + b^2}\sqrt{1 + a'^2 + b'^2}}.$$

The $\cos V$ will be plus or minus, according as we take the signs of the radical factors in the denominator, like or unlike.

The plus value of $\cos V$ will correspond to the acute angle, and the minus value, to the obtuse angle.

1. If we make, $v = 90°$, $\cos V = 0$, and

$$1 + aa' + bb' = 0,$$

which is the equation of condition, when the two lines are perpendicular to each other in space.

EXAMPLES.

1. What is the distance between two points of which the equations are,

$$x' = 5,\ y' = 5,\ z' = 3;\quad x'' = -1,\ y'' = 0,\ z'' = -5?$$

Ans. 11.18 +

2. Find the equations of a line which shall pass through a point whose co-ordinates are, $x' = 3$, $y' = -2$, and $z' = 0$, and be parallel to a line whose equations are,

$$x = z + 1, \text{ and } y = \tfrac{1}{2}z - 2.$$

Ans. $x = z + 3$, $y = \frac{1}{2}z - 2$.

3. Required the equations of a line passing through the two points whose co-ordinates are,

$$x' = 2, \quad y' = 1, \quad z' = 0,$$

and $$x'' = -3, \quad y'' = 0, \quad z'' = -1.$$

Ans. $x = 5z + 2$, $y = z + 1$.

4. Required the angle included between two lines, whose equations are,

$$\left.\begin{aligned} x &= 3z + 5 \\ y &= 5z + 3 \end{aligned}\right\} \text{ of the 1st,}$$

and

$$\left.\begin{aligned} x &= z + 1 \\ y &= 2z \end{aligned}\right\} \text{ of the 2d.}$$

Ans. 14° 58′.

5. Required the angles which a straight line makes with the co-ordinate axes, its equations being,

$$x = -2z + 1,$$
$$y = z + 3.$$

Ans. $\begin{cases} 144^\circ\ 44' \text{ with } X, \\ 65^\circ\ 54' \text{ with } Y, \\ 65^\circ\ 54' \text{ with } Z. \end{cases}$

6. Having given the equations of two straight lines,

$$\left.\begin{aligned} x &= 2z + 1 \\ y &= 2z + 2 \end{aligned}\right\} \text{ of the 1st,}$$

and

$$\left.\begin{aligned} x &= z + 5 \\ y &= 4z + \beta' \end{aligned}\right\} \text{ of the 2d,}$$

required the value of β' so that the lines shall intersect each other, and the co-ordinates of the point of intersection.

Ans. $\begin{cases} \beta' = -6, \\ x' = 9, \\ y' = 10, \\ z' = 4. \end{cases}$

OF THE PLANE.

24. The EQUATION OF A PLANE is an equation, expressing the relation between the co-ordinates of every point of the plane.

To find the equation of a plane.

25. A line is said to be perpendicular to a plane when it is perpendicular to every line passing through its foot

and lying in the plane: and, conversely, the plane is said to be perpendicular to the line.*

A plane may, therefore, be generated by drawing a line perpendicular to a given line, and then revolving this perpendicular about the point of intersection. If the perpendicular be at right angles to the given line, in all its positions, it will generate a plane surface.

Let $$x = az + \alpha, \qquad y = bz + \beta,$$

be the equations of a given line.

If we designate the co-ordinates of a particular point, by x', y', z', the equations of the line passing through this point, will be,

$$x - x' = a(z - z') \quad . \quad (1.) \qquad y - y' = b(z - z') \quad . \quad (2.)$$

The equations of a second line passing through the same point, of which the co-ordinates are x', y', z', are of the form,

$$x - x' = a'(z - z') \quad . \quad . \quad . \quad . \quad (3.)$$

$$y - y' = b'(z - z') \quad . \quad . \quad . \quad . \quad (4.)$$

But the two lines will be at right angles to each other, if their equations fulfill the condition (Art. **23**—1).

$$1 + aa' + bb' = 0 \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

If we now attribute to a' and b', all possible values that will satisfy this equation, we shall have all the perpendiculars which can be drawn to the given line, through the point whose co-ordinates are x', y', z'. These perpendiculars determine the plane.

---

* Legendre, Bk. VI. Def. 1.

It is necessary, however, to find the equation of the plane in terms of the co-ordinates of its different points. We find from Equations (3) and (4),

$$a' = \frac{x - x'}{z - z'}, \qquad b' = \frac{y - y'}{z - z'}.$$

Substituting these values in Equation (5),

$$1 + aa' + bb' = 0,$$

and reducing, we find,

$$z - z' + a(x - x') + b(y - y') = 0;$$

but, since a, b, z', x', y', are known quantities, we may denote the constant part of the equation by a single letter, by making,

$$- z' - ax' - by' = - c.$$

hence, the equation of the plane becomes,

$$z + ax + by - c = 0.$$

1. Since the equation of a plane contains three variables, we may assign values, at pleasure, to two of them, and the equation will then make known the value of the third. For example, if we assign known values, denoted by x' and y', to x and y, the equation of the plane will give,

$$z = c - ax' - by',$$

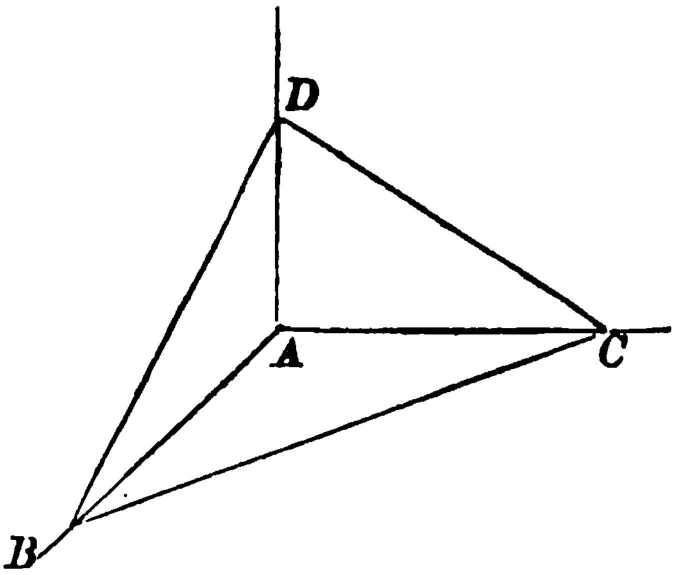
and hence, the co-ordinate z becomes known.

Traces of planes.

26. The lines in which a plane intersects the co-ordi-

nate planes, are called the *traces of the plane*. These traces are found by combining the equation of the plane with the equations of the co-ordinate planes.

Thus, if in the equation,



$$z + ax + by - c = 0,$$

we make $y = 0$, which is the characteristic of the co-ordinate plane ZX, the resulting equation,

$$z + ax - c = 0,$$

will designate the trace CD, common to the two planes. The equation may be placed under the form,

$$z = -ax + c;$$

and hence, the trace may be drawn. Or, if we make, in succession,

$$x = 0, \qquad \text{and} \qquad z = 0,$$

we shall find,

$$z = c = AD, \qquad \text{and} \qquad x = \frac{c}{a} = AC,$$

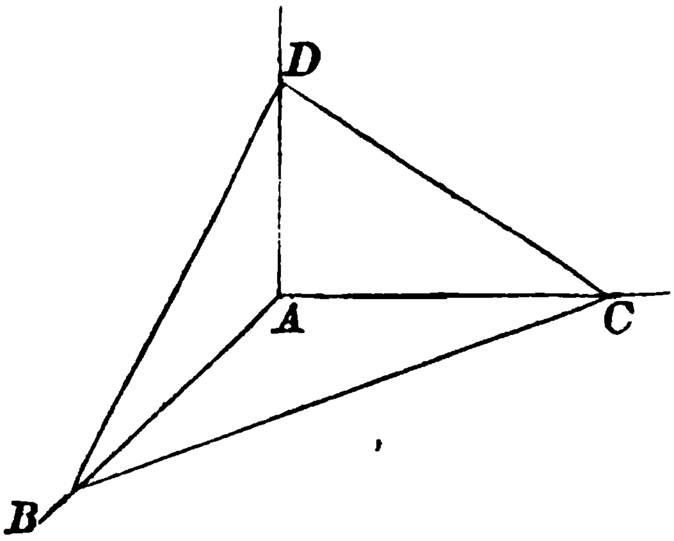
and the trace may then be drawn through the points C and D.

1. We likewise find, for the trace BD,

$$z = -by + c;$$

and for the trace BC, $y = -\frac{a}{b}x + \frac{c}{b}$.

We also find $AD = c$, by making $y = 0$, in the



equation of the trace BD, and $AB = \frac{c}{b}$, by making $x = 0$, in the equation of the trace BC, or by making $z = 0$, in the equation of the trace BD.

2. By comparing the equations of the traces with the equation of a straight line, in Bk. I., Art. **13**, we see that,

$-a$, is the tangent of the angle which the trace CD makes with the axis of X;

$-b$, the tangent of the angle which the trace BD makes with the axis of Y; and

$-\frac{a}{b}$, the tangent of the angle which the trace BC makes with the axis of X.

3. The equation of a plane may be written under the form,

$$Ax + By + Cz + D = 0,$$

in which A, B, C, and D, are constant for the same plane, but have different values when the equation represents different planes. The coefficients A, B, and C, are arbitrary functions of the angles which the traces of the plane form with the co-ordinate axes, and D is an arbitrary function of the distances from the origin to the points in which the plane cuts the co-ordinate axes. If the plane passes through the origin of co-ordinates, its equation takes the form,

$$Ax + By + Cz = 0.$$

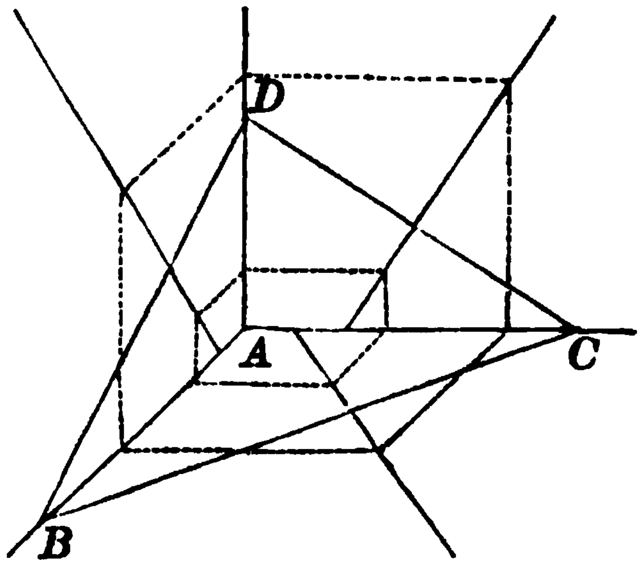
Line perpendicular to a plane.

27. The equations of the straight line, to which the plane has been drawn perpendicular, are,

$$x - x' = a(z - z'), \qquad y - y' = b(z - z');$$

and the equations of the traces CD, BD, may be placed under the form,

$$x = -\frac{1}{a}z + \frac{c}{a}, \quad y = -\frac{1}{b}z + \frac{c}{b}.$$



By comparing the coefficient of z, in the equation of the projection of the line on the co-ordinate plane ZX, with the coefficient of z in the equation of the trace CD, we find, that their product plus unity is equal to 0; hence, the lines are at right angles to each other. The same may be shown for the trace BD, and the projection of the line on the plane YZ; and also for the trace BC, and the projection on the plane YX.

Hence, this property, viz.: *If a line be perpendicular to a plane in space, the projections of the line will be respectively perpendicular to its traces.*

EXAMPLES.

1. Find the traces of a plane whose equation is,

$$z - 9y + 11x - 12 = 0.$$

2. Find the traces of a plane perpendicular to a line, whose equations are,

$$x = 3z + 5, \qquad \text{and} \qquad y = -2z - 4.$$

3. Find the traces of a plane whose equation is,

$$2x - 3y - z = 0.$$

SURFACES OF THE SECOND ORDER.

28. The EQUATION OF A SURFACE, is an equation expressing the relation between the co-ordinates of every point of the surface.

It has been shown (Bk. I., Art. **18**), that every equation of the first degree, between two variables, represents a straight line; and in (Bk. V., Art. **40**—6), that every equation of the second degree, between two variables, represents a circle, an ellipse, a parabola, or an hyperbola.

It has also been shown (Art. **26**—3), that an equation of the first degree between three variables represents a plane, and analogy would lead us to infer what may be rigorously proved, viz.: that *every equation of the second degree, between three variables, represents a curved surface.*

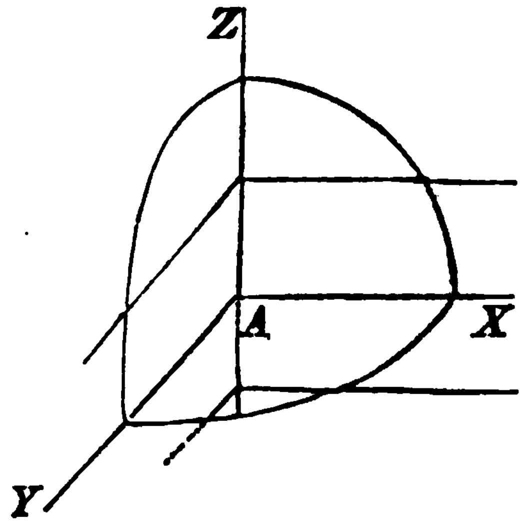
29. Surfaces, like lines, are classed according to the degree of their equations. The plane, whose equation is of the first degree, is a surface of the *first order;* and every surface whose equation is of the second degree, is a surface of the *second order.*

30. The equation of a surface, is an equation which expresses the relation between the co-ordinates of every point of the surface. Although the equation determines the surface, yet it does not readily present to the mind, its form, its dimensions, and its limits. To enable us to conceive of these, we intersect the surface by a system of planes, parallel to the co-ordinate planes. If then, we combine

the equations of these planes with the equation of the surface, the resulting equations will represent the curves in which the planes intersect the surface. These curves will indicate the form, the dimensions, and the limits of the surface.

31. To give a single example, let us take the equation,

$$x^2 + y^2 + z^2 = R^2.$$



Let us intersect the surface represented by this equation, by a plane parallel to YX, and at a distance from it equal to c. The equation of the plane will be (Art. **11**),

$$z = \pm c.$$

Combining this with the equation of the surface, we shall have,

$$x^2 + y^2 = R^2 - c^2,$$

which is the equation of the curve of intersection. This equation represents the circumference of a circle, whose centre is in the axis of Z, and radius, $\sqrt{R^2 - c^2}$. The radius will be real, for all values of c less than R, whether c be plus or minus. It is zero, when c is equal to R, and imaginary, when c is greater than R. Thus, in the first case, the intersection will be the circumference of a circle, in the second case, it will be a point, and in the third, it will be an imaginary curve; or, in other words, the plane will not intersect the surface.

Since the given equation is symmetrical with respect

to the three variables x, y, and z, we may obtain similar results by intersecting the surface by planes parallel to the co-ordinate planes, YZ and ZX.

The co-ordinate planes intersect the surface in circles, whose equations are,

$$x^2 + y^2 = R^2, \qquad x^2 + z^2 = R^2, \qquad y^2 + z^2 = R^2,$$

These results indicate that the surface whose equation is,

$$x^2 + y^2 + z^2 = R^2,$$

is the surface of a sphere; but, to prove it rigorously, it would be necessary to show, that *every* secant plane would intersect it in the circumference of a circle.

Surfaces of revolution.

32. Every surface which can be generated by the revolution of a line about a fixed axis, is called a *surface of revolution.*

The revolving line is called the *generatrix;* the line about which it revolves, is called the *axis of the surface*, or the *axis of revolution.*

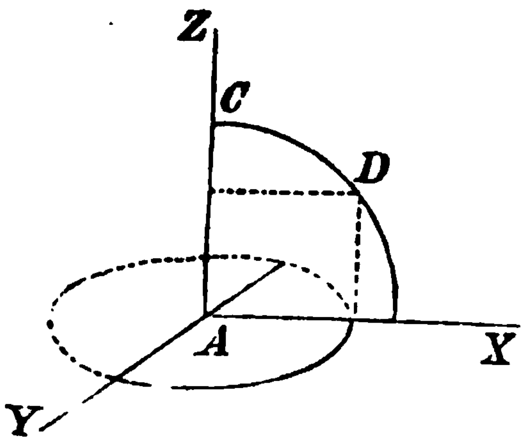
In all the cases considered, we shall suppose the generatrix, in its first position, to be in the co-ordinate plane ZX, and to be revolved about the axis of Z.

33. When the generatrix is a straight line, and not perpendicular to the axis of Z, the surface described is called a surface of *single curvature.* When the generatrix is a curve, the surface is called a surface of *double curvature.*

The section made by a plane passing through the axis, is called a *meridian section;* or a *meridian curve*, when the surface is of double curvature.

34. It is plain, from the definition of a surface of revolution, that every point of the generatrix will describe the circumference of a circle, the centre of which is in the axis of revolution.

35. Let DC be any curve, in the co-ordinate plane ZX, and let it be revolved around the axis of Z; it is required to determine the equation of the surface which it will describe.



If we designate the abscissa of any point of the generatrix, as D, by r, and the ordinate by z, the equation of the generatrix may be written under the form,

$$r = F(z); \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

the value of r is always known, in terms of z and constants, when the equation of the generatrix is given.

We have now to express, analytically, the conditions which will cause this point of the generatrix to describe the circumference of a circle around the axis of Z. To do this, we have only to consider, that the circumference described by the point D, will be projected, on the co-ordinate plane YX, into an equal circumference. If the co-ordinates of the points of this circumference be designated by x and y, we shall have,

$$r = \sqrt{x^2 + y^2} \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

If we now suppose r to take all possible values that will satisfy the equation of the generatrix,

$$r = F(z),$$

and then combine this equation with Equation (2), we shall have,

$$r = F(z) = \sqrt{x^2 + y^2}, \quad . \quad . \quad . \quad (3.)$$

which is the equation of the surface of revolution.

An examination of the construction of Equation (3), will indicate the method of applying it, in finding the equation of any surface.

Equation (1) is the equation of the generatrix from which r is found, in terms of z and constants. This confines the point D, whose co-ordinates are r and z, to the generatrix. Equation (2) requires, that the curve traced by D, be the circumference of a circle; hence, the combination of those two equations, gives the equation of the surface.

36. As a first application, let it be required to find the surface generated by the semi-circumference of a circle, whose centre is at the origin of co-ordinates.

The equation of the generatrix will be,

$$r^2 + z^2 = R^2;$$

hence, $$r = \sqrt{R^2 - z^2}.$$

Substituting this value of r, in Equation (3), we have,

$$\sqrt{R^2 - z^2} = \sqrt{x^2 + y^2};$$

or, $$x^2 + y^2 + z^2 = R^2,$$

which is the equation of the surface of the sphere, when the centre is at the origin of co-ordinates.

37. The volume described by the revolution of an ellipse about either axis, is called, an *ellipsoid of revolution.* It

is also, sometimes, called a *spheroid.* It is called a *prolate spheroid*, when the ellipse is revolved about its transverse axis, and an *oblate spheroid*, when it is revolved about the conjugate axis.

38. Let it be required to find the equation of the surface of a prolate spheroid. If the transverse axis of the ellipse coincides with the axis of Z, the equation of the generatrix will be,

$$B^2z^2 + A^2r^2 = A^2B^2;$$

hence,
$$r = \sqrt{\frac{A^2B^2 - B^2z^2}{A^2}}.$$

Substituting this value, in the general equation of the surface of revolution, Equation (3), we obtain,

$$B^2z^2 + A^2(x^2 + y^2) = A^2B^2,$$

which is the equation of the surface of a prolate spheroid.

39. We find, by a similar process, the equation of the surface of the oblate spheroid, to be,

$$A^2z^2 + B^2(x^2 + y^2) = A^2B^2.$$

If in either of these equations, we make $A = B$, we obtain,

$$x^2 + y^2 + z^2 = R^2,$$

the equation of the surface of a sphere.

40. If an hyperbola be revolved about its transverse axis, each branch will describe a volume. The surface of each volume is called a *nappe*, and the two volumes taken

together, are called, an *hyperboloid of revolution of two nappes.*

The volume described by the revolution of an hyperbola about its conjugate axis, is called, an *hyperboloid of revolution of one nappe.*

41. If the transverse axis of an hyperbola coincides with the axis of Z, the centre being at the origin, its equation will be,

$$A^2x^2 - B^2z^2 = -A^2B^2.$$

If the conjugate axis coincides with Z, we have,

$$A^2x^2 - B^2z^2 = A^2B^2 \quad \text{(Bk. V., Art. 11).}$$

In the first case, the equation of the surface is,

$$B^2z^2 - A^2(x^2 + y^2) = A^2B^2; \text{ in the second,}$$

$$A^2z^2 - B^2(x^2 + y^2) = -A^2B^2.$$

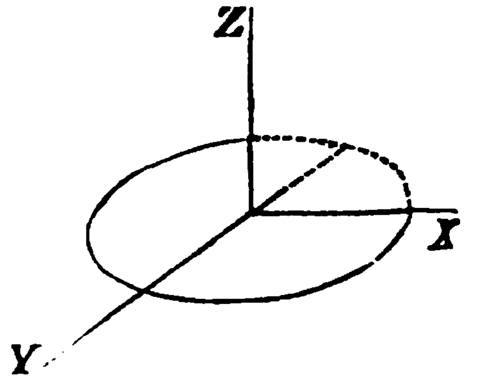
42. If a parabola be revolved around its axis, the volume described, is called a *paraboloid of revolution.* The equation of the generatrix being,

$$r^2 = 2pz,$$

the equation of the surface will be,

$$x^2 + y^2 = 2pz.$$

Surfaces of single curvature.



43. Let the generatrix be a straight line parallel to the axis of Z. Equation (1) will then become,

$r = F(z) = a$, an arbitrary constant;

that is, for *every value* of z, r will be *constant.*

Equation (2) will become,

$$r = \sqrt{x^2 + y^2},$$

or, $$x^2 + y^2 = r^2;$$

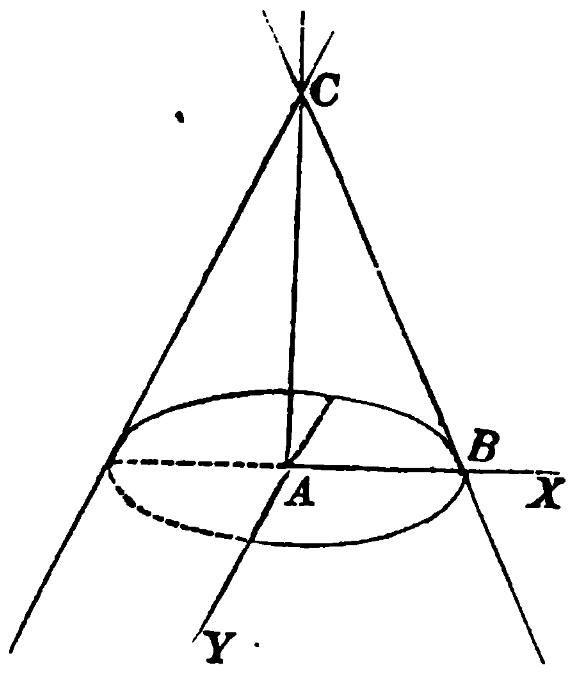
hence, the equations of the surface, are,

$$r = \text{an arbitrary constant}, \quad \text{and} \quad x^2 + y^2 = r^2.$$

The first condition indicates, that every point of the generatrix is at the same distance from the axis of Z; hence, each point will describe an equal circumference. The second condition indicates, that all these circumferences will be projected into the same circumference, on the co-ordinate plane YX; hence, the surface is that of a *right cylinder with a circular base.*

Surface of the cone.

44. Let the generatrix be any straight line oblique to the axis of Z, as BC. Denote the distance AC, by c. Then, since BC passes through the point C, whose co-ordinates are, $z' = c$, and $x' = 0$, its equation (Bk. I., Art. **20**), is,



$$z = ax + c;$$

if r denote the abscissa of any point of the generatrix, and z the ordinate, its equation (Art. **35**) will be,

$$z = ar + c; \qquad \text{whence,} \qquad r = \frac{z - c}{a};$$

whence, we have (Art. 35),

$$\frac{z-c}{a} = \sqrt{x^2+y^2};$$

or,
$$(z-c)^2 = a^2(x^2+y^2) \quad . \quad . \quad . \quad (1.)$$

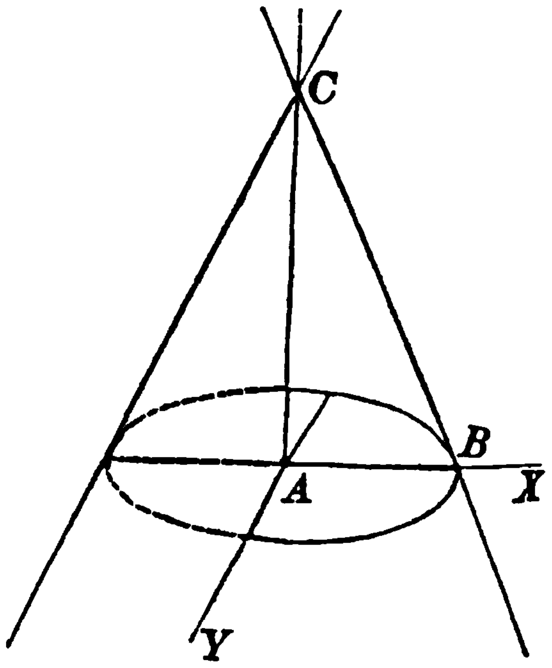
In this equation, a is the tangent of the angle CBX. Denote its supplement, CBA, by v; then,*

$$a = -\tan v; \qquad \text{or,} \qquad a^2 = \tan^2 v;$$

hence, Equation (1) becomes,

$$(x^2+y^2)\tan^2 v = (z-c)^2 \quad . \quad . \quad . \quad (2.)$$

This is the equation of the surface of the cone generated by the line BC, revolving about the axis Z. It is a *right cone*, with a circular base. C is the vertex of the cone, AB, the radius of the circle in which it is intersected by the co-ordinate plane YX, and v, the angle which the generatrix makes with the base.



If the generatrix, BC, be prolonged beyond the point C, the prolongation will generate an equal conical surface, lying above the vertex C. The conical surface below the point C, is called the *lower nappe* of the cone, and the surface above C, the *upper nappe* of the cone.

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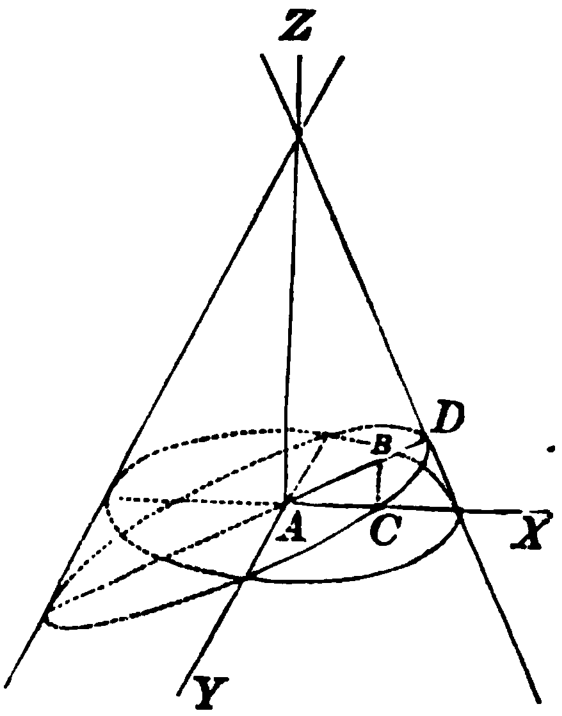
* Legendre, Trig. Art. 25.

45. Let the surface of this cone be now intersected by a plane, passing through the axis of Y, and consequently perpendicular to the co-ordinate plane ZX. Designate by u, the angle DAX, which the secant plane makes with the co-ordinate plane YX. The equation of this plane will be the same as that of its trace AD (Art. **18**—1); that is,



$$z = x \tan u.$$

If we combine this equation with the equation of the surface, we shall obtain the equation of their curve of intersection. This equation is of the simplest form, when the curve is referred to two axes in its own plane. Let us, then, refer it to the two axes, AY, AD, in the plane of the curve, and at right angles to each other.

If we designate the co-ordinates of any point, referred to these axes, the one, for example, which is projected at B, by x', y', we shall have,

$$AC = x = x' \cos u, \qquad BC = z = x' \sin u;$$

and since the axis of Y is not changed,

$$y = y'.$$

Substituting these values in equation of the surface, (Equation 2), we shall obtain, after reduction, the equation of intersection,

$$y'^2 \tan^2 v + x'^2 \cos^2 u (\tan^2 v - \tan^2 u) + 2cx' \sin u = c^2;$$

or, omitting the accents,

$$y^2 \tan^2 v + x^2 \cos^2 u (\tan^2 v - \tan^2 u) + 2cx \sin u = c^2.$$

This equation is of the same form as Equation (4), Bk. V., Art. **38**; hence, what was proved of that, may be proved of this. Therefore,

1st. When the coefficients of y^2 and x^2, have the same sign, the curve will be an ellipse:

2d. When the coefficient of x^2, is zero, the curve will be a parabola:

3d. When the coefficients of y^2 and x^2, have unlike signs, the curve will be an hyperbola.

Since the $\tan^2 v$ is always positive, the change of signs in the coefficients of y^2 and x^2, must arise from the change of sign in the coefficient of x^2; and since $\cos^2 u$ is positive, the sign of this coefficient will depend on the relative values of v and u. When $v > u$, it will be positive; when $v = u$, it will be zero; when $u > v$, it will be negative.

46. In order to obtain the forms and classes of the curves which result from the intersection of the cone and plane, it might, at first, seem necessary to cause the angle u to vary from 0 to 360°. But since the surface of the cone is symmetrical with respect to its axis, it is plain that all the varieties will be obtained by varying u from 0 to 90°.

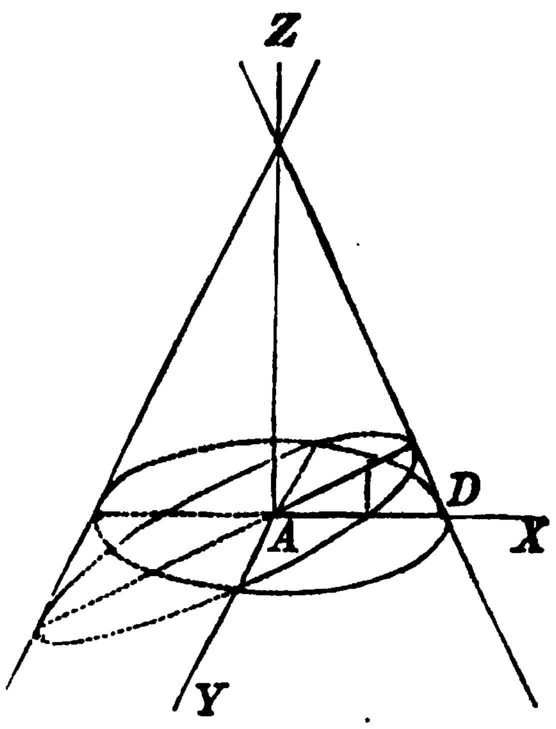
47. Let us then resume the equation of intersection,

$$y^2 \tan^2 v + x^2 \cos^2 u (\tan^2 v - \tan^2 u) + 2cx \sin u = c^2,$$

and begin the discussion of it, by supposing,

$$u = 0,$$

which will cause the secant plane to coincide with the co-ordinate plane YX. The equation of the curve will then become,



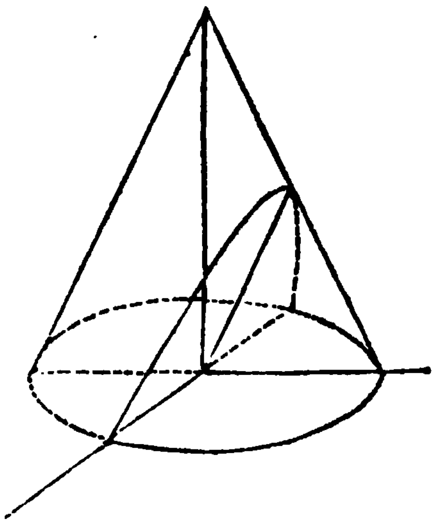
$$x^2 + y^2 = \frac{c^2}{\tan^2 v};$$

hence, the curve is the circumference of a circle, of which A is the centre, and AD equal to $\dfrac{c}{\tan v}$, the radius.

48. If we now suppose u to increase, the curve of intersection will be an ellipse, so long as $u < v$; that is,

If a right cone with a circular base be intersected by a plane, making with the base of the cone an angle less than the angle formed by the generatrix and base, all the elements of the same nappe will be intersected, and the curve of intersection will be an ellipse.

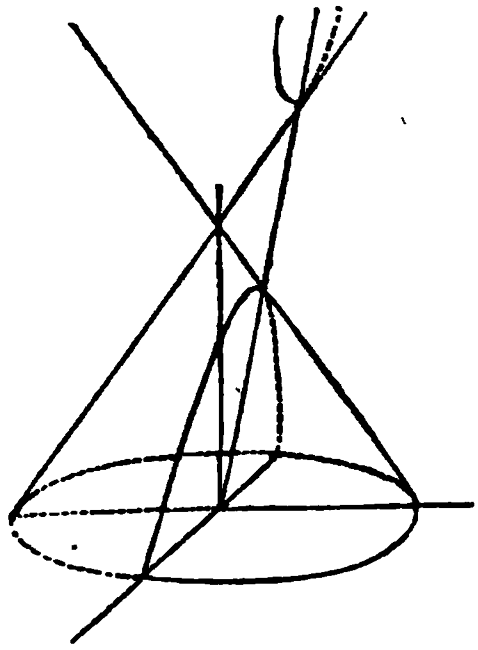
49. When u becomes equal to v, the cutting plane becomes parallel to a generatrix of the cone: hence,



If a right cone with a circular base be intersected by a planc

parallel to the generatrix, the curve of intersection will be a parabola.

50. When u becomes greater than v, the cutting plane will intersect both nappes of the cone; hence,



If a right cone with a circular base be intersected by a plane making with the base of the cone an angle greater than the angle formed by the generatrix and base, both nappes of the cone will be intersected, and the curves of intersection will be an hyperbola.

DIFFERENTIAL

AND

INTEGRAL CALCULUS,

DESIGNED FOR

ELEMENTARY INSTRUCTION.

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PREFACE.

THE DIFFERENTIAL AND INTEGRAL CALCULUS is too important a part of Mathematical Science to be entirely omitted in a course of Collegiate instruction.

The abstract quantities, Number and Space, are presented to the mind, in the elementary branches of Mathematics, as of definite extent, and as made up of parts; and the value or measure (how much?) in any given case, is expressed by the number of times which the quantity contains one of its parts, regarded as a unit of measure. But we do not attain to a clear apprehension of their *quantitative nature*, until we regard them as of indefinite extent, as possessing continuity, and as capable of changing from one state of value to another, according to any conceivable law.

The Differential and Integral Calculus embraces all the processes necessary to such an analysis. It regards quantity as the result of change. It examines established laws of change and determines their consequences. It supposes

laws of change and traces the results of the hypothesis. In short, it embraces within its grasp—in the Material, everything from the minuest atom to the largest body—in Space, all that can be measured, from the geometrical point to absolute infinity—in Time, the entire range of duration—and in Motion, every change from absolute rest to infinite velocity.

The German, the French, the English, and even the American press, has been prolific in the number of Treatises recently published on this subject. The effort to furnish better Text-Books, proves at once the value of the knowledge, and the great difficulty of presenting it in the best possible form.

In regard to the Treatise now presented to the public, I have simply to say, that it is an Elementary Text-Book for the use of College Classes, and other classes of about the same grade. The Treatise of Professor COURTENAY, late of the Virginia University, and that of Professor CHURCH, of the Military Academy, may be advantageously read by those who may wish to advance further; and it is due to Columbia College to state that this Treatise is used in the Course prescribed to all the pupils, and is not an exponent of the higher course pursued by those who make Mathematical Science a special branch.

COLUMBIA COLLEGE, *June*, 1860.

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INTRODUCTION.

COMMON TERMS must always be employed in definitions, because a definition refers to a class of things in which each enjoys at least one property common to all the others. Each individual of a class, so defined, is called a significate. A common term does not express to the mind a distinct and adequate idea of any one of its significates, but a general notion of them all; hence, we do not comprehend the full scope and meaning of a definition, until we have ascertained, by careful analysis, the number of its significates and the exact characteristics of each.

MATHEMATICS is the science which treats, *primarily*, of the relations and measures of quantities; and secondarily, of the operations and processes by which these relations and measures are ascertained.

QUANTITY is anything that can be increased, diminished, and measured. There are two general kinds, Number and Space, and each is subdivided into four classes. Under Number, we have Abstract Number, Currency, Weight, and Time; and under Space, Length, Surface, Volume, and Angular Measure.

MATHEMATICS, considered as a science of exact relation, is divided into three branches: 1. Arithmetic. 2. Geometry. 3. Analysis.

Arithmetic is that branch which treats of the properties and relations of Numbers, when expressed by figures.

Geometry treats of the properties and relations of Magnitudes, by reasoning directly on the magnitudes themselves, or upon their pictorial representations. The magnitudes considered in this branch of Mathematics, are, lines, surfaces, volumes, and angles.

Analysis embraces all that portion of Mathematics in which the quantities considered are represented by letters, and the operations to be performed are indicated by signs, or conventional symbols. Its elementary branches are, Algebra, Analytical Geometry, and Analytical Trigonometry. In these branches, quantities of the same kind are compared by means of their unit of measure, which has a fixed value, and is generally expressed, numerically, by the unit 1.

A Variable Quantity is one which increases or diminishes, according to any law, and thus passes from one state of value to another. If, in changing its value, between any two limits, it passes through all the intermediate values, it is called a *continuous quantity*.

If we suppose a variable quantity, denoted by x, to have a particular value, $x = a$, and afterwards to assume another value, $x = a'$, we may suppose that x changes *uniformly* from a to a', and assumes, in succession, all the values between its limits. These two suppositions render it impossible to express the change in value, either by 1, or by any of the parts of 1.

For, denote the uniform change in x, by h: then, if h could be expressed by a fraction, however small, that fraction could be diminished by increasing its denominator:

hence, there would be values between a and a', through which x would not pass, which is contrary to the hypothesis. Therefore, the hypothesis that x changes uniformly, and passes through all values between $x = a$ and $x = a'$, renders it impossible to express the change of value by numbers.

When we say that a continuous quantity passes from one state of value to another, we mean that it either increases or diminishes; and when we speak of the *next* value, we mean the first value which it assumes when a change begins. These two values are called *consecutive values*, whether the change be uniform or varying. When the change is uniform, we have seen that the difference between consecutive values cannot be expressed in 1, or in parts of 1. The same is also true when the change is not uniform. For, if the difference of two consecutive values could be expressed in 1, or in parts of 1, it could be diminished; hence, there would be intermediate values, which is contrary to the definition.

The hypothesis, therefore, of continuous quantity, renders it impossible to express the elementary changes of value by means of numbers; and hence, we are unable to deal with such changes by any of the methods already explained.

The Calculus is the name given to that branch of Mathematics which treats of the properties and relations of continuous quantities—of the laws of change to which they may be subjected, and the results flowing from such changes.

When we measure a quantity, great or small, the standard or unit of measure is of the same kind as the quantity

measured, and the ratio of this unit to the quantity is the result of the measurement. In the operations of the Calculus, the unit of measure is the change which takes place in the quantity that varies uniformly. This quantity is called the *independent variable.* The quantity whose changes of value are measured, is called the *function.* The independent variable and function are connected by a law, either expressed or implied, and change simultaneously according to that law.

The Theory of the Calculus, therefore, rests on the following axioms and inferences:

1. Where no law of change has been fixed, such a law may be imposed as will cause the variable to change uniformly, and to pass through all values between any two limits.

2. The difference between any two consecutive values of a quantity so varying, is constant.

3. A quantity changing its rate according to this law furnishes a standard, or unit of measure, by means of which the changes in all other variable quantities may be relatively determined.

4. When a variable quantity changes from one state of value to another, there exists a difference between two consecutive values, which forms no appreciable part of the quantity itself.

5. The ratio of the change in the independent variable, to the corresponding change of the function, is the *rate* of change of the function; and the actual change is denoted by the difference between two consecutive values.

DIFFERENTIAL CALCULUS.

SECTION I.

DEFINITIONS AND FIRST PRINCIPLES.

Definitions.

1. In the Differential Calculus, as well as in Analytical Geometry, the quantities considered are divided into two classes:

1st. *Constant quantities, which preserve the same values in the same investigation;* and,

2d. *Variable quantities, which assume all possible values that will satisfy any equation which expresses the relation between them.*

The constants are denoted by the first letters of the alphabet, a, b, c, &c.; and the variables, by the final letters, x, y, z, &c.

Uniform and varying changes.

2. There are two ways in which a variable quantity may pass from one value to another.

If the variable x, once had the particular value, $x = a$, and afterwards assumed the value, $x = a'$, we can suppose:

1st. That during the change from a to a', x assumed, in succession, and by a uniform change, *all the values* between a and a', just as a body moving uniformly over a given straight line passes through all the points between its extremities; or,

2d. We may suppose, that during the change from a to a', x assumed all possible values between its limits, without the condition of a uniform change. In both cases, the quantity is said to be *continuous.*

3. If two variable quantities, y and x, are connected in an equation, as, for example,

$$y = x^2 + 2;$$

then, to every value of x, arbitrarily assumed, there will be a corresponding value of y, *dependent upon, and resulting from,* the value attributed to x. Thus, if we make $x = 4$, we have,

$$y = 16 + 2 = 18.$$

If we suppose x to increase from 4 to 5, we shall have,

$$y = 25 + 2 = 27;$$

thus, while x changes from 4 to 5, y changes from 18 to 27.

If now we suppose x to increase from 5 to 6, y will increase from 27 to 38. Thus, while x increases uniformly by 1, y will change its value according to a very different law.

Function and variable.

4. When two variable quantities, y and x, are connected in an equation, either of them may be supposed

to increase or decrease uniformly; such variable is called the *independent variable*, because the *law of change is arbitrary*, and *independent* of the form of the equation. This variable is generally denoted by x, and called simply, the *variable*. The change in the variable y, depends on the *form* of the equation; hence, y is called the *dependent* variable, or *function*. When such a relation exists between y and x, it is expressed by an equation of the form,

$$y = F(x), \qquad y = f(x); \qquad \text{or,} \qquad f(y, x) = 0;$$

which is read, y a function of x. The letter F, or f, is a mere symbol, and stands for the word, *function*. If y is a function of x, that is, changes with it, x is also a function of y; hence,

One quantity is a function of another, when the two are so connected that any change of value, in either, produces a corresponding change in the other.

5. If the equation connecting y and x, is of such a form that y occurs *alone*, in the first member, y is called an *explicit* function of x. Thus, in the equations,

$$y = ax + b \quad . \quad . \quad . \quad . \quad . \quad \text{of a straight line,}$$

$$y = \sqrt{R^2 - x^2} \quad . \quad . \quad . \quad . \quad \text{of the circle,}$$

$$y = \frac{B}{A}\sqrt{A^2 - x^2} \quad . \quad . \quad . \quad \text{of the ellipse,}$$

$$y = \sqrt{2px} \quad . \quad . \quad . \quad . \quad . \quad \text{of the parabola, and}$$

$$y = \frac{B}{A}\sqrt{x^2 - A^2} \quad . \quad . \quad . \quad \text{of the hyperbola,}$$

y is an *explicit* function of x.

But, if the equations are written under the forms,

$$y - ax - b = 0; \quad \text{or,} \quad f(x, y) = 0,$$
$$y^2 + x^2 - R^2 = 0; \quad \text{or,} \quad f(x, y) = 0,$$
$$A^2y^2 + B^2x^2 - A^2B^2 = 0; \quad \text{or,} \quad f(x, y) = 0,$$
$$y^2 - 2px = 0; \quad \text{or,} \quad f(x, y) = 0,$$
$$A^2y^2 - B^2x^2 + A^2B^2 = 0; \quad \text{or,} \quad f(x, y) = 0,$$

y is called an *implicit* function of x; the nature of the relation between y and x being *implied*, but not developed in the equation.

6. It is plain, that in each of the above equations the *absolute* value of y, for any given value of x, will depend on the constants which enter into the equation; this relation is expressed, by calling y an *arbitrary* function of the constants on which it depends. Thus, in the equation of the straight line, y is an arbitrary function of a and b; in the equation of the circle, y is an arbitrary function of R; in the equation of the ellipse, of A and B; in the equation of the parabola, of $2p$; and in the equation of the hyperbola, of A and B.

7. An *increasing* function is one which increases when the variable increases, and decreases when the variable decreases. A *decreasing* function is one which decreases when the variable increases, and increases when the variable decreases.

In the equation of a straight line, in which a is positive, y is an increasing function of x. In the equations of the circle and ellipse, y is a decreasing function of x. In the equation of the parabola, y is an increasing function of x. In the equation of the hyperbola, y is imaginary

for all values of $x < A$, and an increasing function for all positive values of $x > A$.

8. A quantity may be a function of two or more variables. If

$$u = ax + by^2, \quad \text{or} \quad u = ax^2 - by^3 + cz + d,$$

u will be a function of x and y, in the first equation, and of x, y, and z, in the second. These expressions may be thus written:

$$u = f(x, y), \quad \text{and} \quad u = f(x, y, z).$$

If, in the second equation, we make, in succession, the independent variables x, y, and z, respectively equal to 0, we have,

for, $x=0$, $\quad u = -by^3+cz+d = f(y, z)$,
for, $x=0$, and $y = 0$, $\quad u = cz + d = f(z)$; and,
for, $x=0$, $y=0$, and $z = 0$, $\quad u = d =$ a constant.

Algebraic and Transcendental Functions.

9. There are two general classes of functions: *Algebraic* and *Transcendental.*

Algebraic functions are those in which the relation between the function and the variable can be expressed in the language of Algebra alone: that is, by addition, subtraction, multiplication, division, the formation of powers denoted by constant exponents, and the extraction of roots indicated by constant indices.

Transcendental functions are those in which the relation between the function and variable cannot be expressed in the language of Algebra alone. There are three kinds:

1. *Exponential* functions, in which the variable enters as an exponent; as,

$$u = a^x.$$

2. *Logarithmic* functions, which involve the logarithm of the variable; as,

$$u = \log x.$$

3. *Circular* functions, which involve the arc of a circle, or some function of the arc; as,

$$u = \sin x, \qquad u = \cos x, \qquad u = \tan x.$$

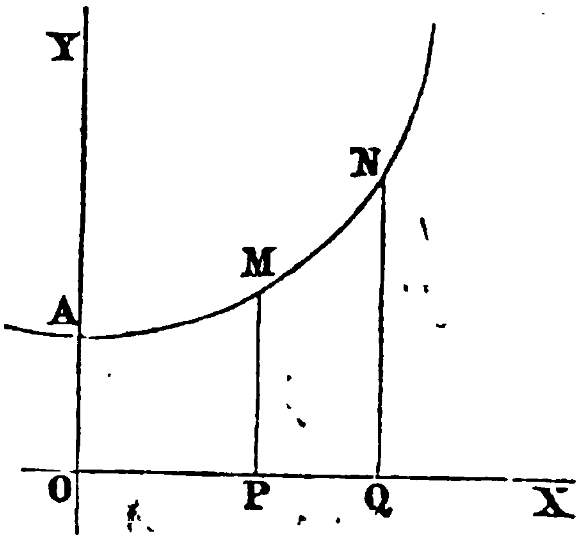
Geometrical representation of Functions.

10. With the aid of Analytical Geometry, it is easy to trace, geometrically, the numerical relation between any function and its independent variable.

Suppose we have given the equation,

$$y = f(x).$$

If we attribute to x, the independent variable, in succession, every value between $-\infty$ and $+\infty$, each will give a corresponding value for y, which may be determined from the equation, $y = f(x)$



Let O be the origin of a system of rectangular co-ordinates. From O, lay off to the right, all the positive values of x, and to the left all the negative values. Through the extremity of each abscissa, so determined, draw a line parallel to the axis of ordinates, and equal to the corresponding value of y; the plus values

will fall above the axis of X, and the negative values below it; then trace a curve, AMN, through the extremities of these ordinates. The co-ordinates of this curve will indicate every relation between y and x, expressed by the equation,

$$y = f(x).$$

This curve should present to the mind, not merely any particular value of x, and the corresponding value of y, but the entire *series of corresponding values* of these two variables.

Language of numbers inadequate.

11. Suppose now, that we give to x a particular value, denoted by OP. To this will correspond a determinate value of y, found from the equation,

$$y = f(x).$$

This value of y will be denoted by MP. Let x, starting from the value OP, increase by a quantity denoted by h, and represented by PQ. The function y will change, in consequence, to some new value denoted by y' and represented by QN. Since the ordinate $y = MP$, is represented under the form,

$$y = f(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

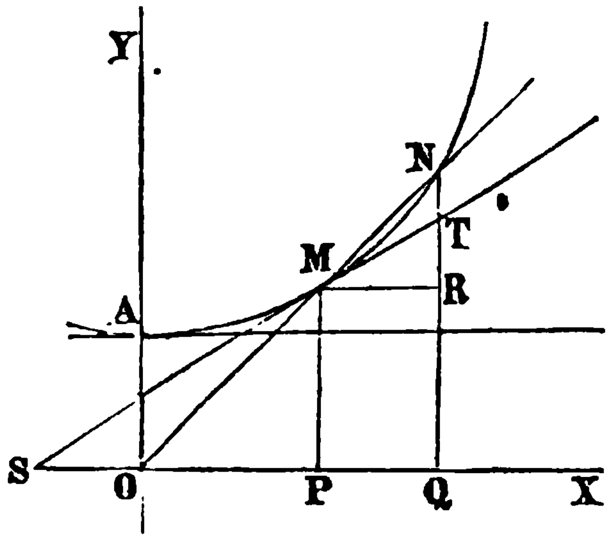
the new ordinate NQ will be expressed under the form,

$$y' = f(x + h) \; . \quad . \quad . \quad . \quad . \quad (2.)$$

If we subtract Equation (1) from (2), we obtain,

$$y' - y = f(x + h) - f(x) \quad . \quad . \quad (3.)$$

It is evident, that each member of this equation will reduce to 0, when we make $h = 0$.



If the abscissa x, in increasing from P to Q, passes through all the values of the abscissas between the limits OP and OQ, then, h will pass through all values between 0 and PQ. If the change in h is *uniform*, and such as to embrace *every value* between 0 and any arbitrary limit, denoted by PQ, it is plain that such change cannot be expressed by a number of which the unit is 1, or any part of 1. For, however small may be the decimal fraction denoting the change, that fraction might still be diminished by prefixing ciphers; hence, there would be values through which h would not pass, which is contrary to the supposition. Therefore, the hypothesis, that x *changes uniformly*, and passes through *all values* between the limits $x = OP$, and $x = OQ$, renders it impossible to express the change of value by numbers. A new language, therefore, becomes necessary.

Consecutive values and Differentials.

12. Let us suppose the point N, of the curve, to approach the point M. Under this supposition, the point Q will approach the point P, and h will diminish towards 0.

If N be made to coincide with M, Q will coincide with P; h will become 0, and y' will become equal to y, or $y' - y = 0$. But before N can coincide with M, there will be a *last* value of h; this value is designated by dx, and is read, *differential of* x; the letter d being merely a symbol, and standing for the words "differential of." For *this value* of h, the difference of the ordinates, denoted

by $y' - y$, is designated by dy, which is read, *differential of* y; the letter d, as before, standing for the words, "differential of." Under this supposition, the values of the abscissas, x and $x + dx$, are called *consecutive* values, and so, also, are the values of y' and y; hence,

Two values of a function or variable, are consecutive when they admit of no intermediate value under the same law of change.

The difference between two *consecutive values* cannot be expressed by a number. For, if it could, such number might be diminished, and then there would be intermediate values, which is contrary to the supposition. Therefore, the difference between two consecutive values is, *numerically*, equal to 0.

The DIFFERENTIAL *of a variable quantity is the difference between any two of its consecutive values.*

Differential Coefficient.

13. If we divide both members of Equation (3) by h, we shall have,

$$\frac{y' - y}{h} = \frac{f(x + h) - f(x)}{h} \quad . \quad . \quad . \quad (4.)$$

Having drawn MR parallel to the axis of abscissas, NR will denote the difference of the two ordinates y' and y; hence, the first, and consequently, the second member of Equation (4) will denote the tangent of the angle NMR, or NOQ, which the secant NM makes with the axis of X.* When the ordinates y' and y become consecutive, the secant MN becomes tangent to the curve at the point

* Trig., Art. 30—31.

M, and the angle MOP changes to the angle MSP, which the tangent line to the curve, at the point M, makes with the axis of X. Denote this angle by α. Equation (4), then, takes the form,

$$\frac{dy}{dx} = \tan \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

The first member of the equation is called the *differential coefficient* of y; hence,

The differential coefficient of a function is the differential of the function divided by the differential of the variable.

If we multiply both members of Equation (5), by dx, we shall have,

$$\frac{dy}{dx} dx = \tan \alpha \times dx.$$

Before y' and y became consecutive, we had,

$$y' - y = NR = \tan NMR \times h;$$

When they become consecutive,

$$dy = \tan \alpha \, dx = \frac{dy}{dx} dx,$$

from the last equation; therefore, $\frac{dy}{dx} dx$ is equal to the differential of the function y; hence,

The differential of a function is equal to its differential coefficient multiplied by the differential of the variable.

Form of the difference between two states of a function.

14. Let us now determine what *form* the second member of Equation (3) assumes, for *any value* of h.

Equation (3) is,

$$y' - y = f(x + h) - f(x) \quad . \quad . \quad . \quad (3.)$$

1. If h be made equal to 0, the first and second members will each reduce to 0. Therefore, after reductions, there will be no term in the second member, that will not contain h, as a factor. Hence, the second member of Equation (3) is divisible by h.

2. After dividing both members of Equation (3) by h, and passing to consecutive values, by making h numerically equal to 0, the second member reduces to tan α, a quantity independent of h (Equation (5); hence, *there is one term in the second member of Equation* (3) *that contains only the first power of h; and the coefficient of this term is tan* α. Since all the other terms become 0, when $h = 0$, each of them must contain h to a higher power than the first. The coefficient of the first power of h, is the differential coefficient of y (Art. **13**).

If we designate by P, the differential coefficient of y, and by P' such a value, that $P'h^2$ shall be equal to all the terms of Equation (3), after the first, Equation (3) may be written under the form,

$$y' - y = Ph + P'h^2 \quad . \quad . \quad . \quad . \quad (6.)$$

Since P, the differential coefficient of y, is equal to the tangent of the angle which the tangent line makes with the axis of abscissas, it will, in general, be a function of the abscissa x, and P' will be a function of x and h.

3. If we have a function of the form,

$$y = ax^3, \quad . \quad . \quad . \quad \quad . \quad . \quad (1.)$$

and give to x an increment h, we then have,

$$y' = a(x+h)^3 = ax^3 + 3ahx^2 + 3ah^2x + ah^3 \quad . \quad (2.)$$

and by subtracting Equation (1) from (2), we have,

$$y' - y = 3ahx^2 + 3ah^2x + ah^3 \quad . \quad . \quad (3.)$$

in which, $P = 3ax^2$, and $P' = 3ax + ah$.

If we divide both members of Equation (3) by h,

$$\frac{y'-y}{h} = 3ax^2 + 3ahx + ah^2;$$

passing to the consecutive values of y and x,

$$\frac{dy}{dx} = 3ax^2, \quad \text{and} \quad \frac{dy}{dx}dx = 3ax^2dx.$$

To find the differential coefficient and the differential.

15. To find the differential coefficient of a function, and also, the differential of the function:

1. *Give to the independent variable an arbitrary increment, and find the corresponding value of the function; from this subtract the primitive function.*

2. *Divide the remainder thus obtained by the increment, and then pass to the consecutive values, by making the increment numerically equal to 0. The result obtained will be the differential coefficient, and this, multiplied by the differential of the variable, gives the differential of the function.*

Equal functions have equal differentials.

16. If two functions, u and v, dependent on the same variable x, are *equal* to each other, for all possible values of x, their differentials will also be equal.

For, x being the independent variable, we have (Art. 14),

$$u' - u = Ph + P'h^2,$$
$$v' - v = Qh + Q'h^2,$$

in which P is the differential coefficient of u, regarded as a function of x, and Q the differential coefficient of v, regarded as a function of x.

But, since u' and v' are, by hypothesis, equal to each other, as well as u and v, we have,

$$Ph + P'h^2 = Qh + Q'h^2,$$

or, by dividing by h and passing to consecutive values,

$$P = Q,$$

hence, $$\frac{du}{dx} = \frac{dv}{dx},$$

and, $$\frac{du}{dx}dx = \frac{dv}{dx}dx,$$

that is, the differential of u is equal to the differential of v.

Converse not true.

17. The converse of this proposition is not generally true; that is,

If two differentials are equal to each other, we are not at liberty to conclude that the functions from which they were derived, are also equal.

For, let $$u = v \pm A \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

in which A is a constant, and u and v both functions of x. Giving to x an increment h, we shall have,

$$u' = v' \pm A,$$

from which subtract Equation (1), and we obtain,

$$u' - u = v' - v,$$

and, by substituting for the difference between the two states of the function, we have,

$$Ph + P'h^2 = Qh + Q'h^2.$$

Dividing by h, and passing to consecutive values, we obtain,

$$P = Q; \quad \text{that is,} \quad \frac{du}{dx} = \frac{dv}{dx};$$

hence, $$\frac{du}{dx}dx = \frac{dv}{dx}dx; \quad \text{or,} \quad du = dv.$$

Hence, the differentials of u and v are equal to each other, although v may be greater or less than u, by any constant quantity A; therefore,

Every constant quantity connected with a variable by the sign plus or minus, will disappear in the differentiation.

The reason of this is apparent; for, a constant does not increase or decrease with the variable; hence, there is no ultimate or last difference between two of its values; and this *ultimate or last difference* is the differential of a variable function. Hence, the differential of a constant quantity is equal to 0.

18. If we have a function of the form,

$$u = Av,$$

in which u and v are both functions of x, and give to x an increment h, we shall have,

$$u' - u = A(v' - v),$$

or, $$Ph + P'h^2 = A(Qh + Q'h^2).$$

Dividing by h, and passing to the consecutive values,

$$P = AQ,$$

or,

$$Pdx = AQdx.$$

But, $du = Pdx$, and $dv = Qdx$;

hence, $du = Adv$; that is,

The differential of the product of a constant by a variable quantity, is equal to the constant multiplied by the differential of the variable.

Signs of the differential coefficient.

19. If u is any function of x, and we give to x an increment h, we have,

$$\frac{u' - u}{h} = P + P'h;$$

and since h is positive, the sign of the first member will be positive when $u < u'$; that is, when u is an *increasing* function of x (Art. 7). It will be negative when $u > u'$; that is, when u is a *decreasing* function of x. Passing to consecutive values, we have, under the first supposition,

$$\frac{du}{dx} = +P; \text{ and}$$

$$\frac{du}{dx} = -P, \text{ under the second; hence,}$$

The differential coefficients of increasing functions are POSITIVE, *and of decreasing functions*, NEGATIVE.

If we multiply by dx, we obtain the differentials, which have the *same signs* as the differential coefficients.

Nature of a differential coefficient, and of a differential.

20. The method of treating the Differential Calculus, adopted in this treatise, is based on three hypotheses:

1st. That the independent variable changes uniformly:

2d. That in changing from one state of value to another, it passes through all the intermediate values; and,

3d. That all functions dependent upon it, undergo changes determined by the equation expressing the relations between them; and that such equations preserve the same general form.

If the independent variable changes uniformly, and assumes all possible values between the limits $x = a$, and $x = a'$, it is plain that the change cannot be denoted by a number (Art. **11**). If, then, we denote this change by dx, we assume that dx is *smaller than any number;* or, that its numerical value is 0; hence,

$$\frac{1}{dx} = \infty.$$

But,
$$\frac{1}{dx} = \frac{dx}{dx^2} = \frac{dx^2}{dx^3} = \frac{dx^3}{dx^4}, \text{ \&c.}$$

that is, any power of dx divided by a power of dx greater by 1, is *infinite*; hence, *any* power of dx is 0, compared with the power next less. Hence, it follows:

1st. That the addition of dx to any number, can make no alteration in its value; and therefore, when connected with a numeral quantity by the sign $\pm$, may be omitted without error; thus,

$$3ax + dx = 3ax.$$

2d. Since dx^2 is 0, compared with dx; that is, *infinitely less than* dx, we have,

$$5ax^2dx + dx^2 = 5ax^2dx;$$

and similarly for the higher powers of dx.

The quantities, dx, dx^2, dx^3, &c., are called *infinitely small quantities, or* INFINITESIMALS *of the first, second, and third orders:* from their law of formation, it follows that,

Every infinitely small quantity may be omitted without error when connected by the sign ± with any of a lower order.

Rate of change.

21. The measure of a quantity, great or small, is the number of times which it contains some other quantity of the same kind, regarded as a unit of measure.

In the Differential Calculus, dx, the differential of the independent variable, is the unit of measure. The *rate of change*, in the function y, is therefore expressed by $\frac{dy}{dx}$, and the actual change corresponding to dx, by

$$\frac{dy}{dx}dx = dy.$$

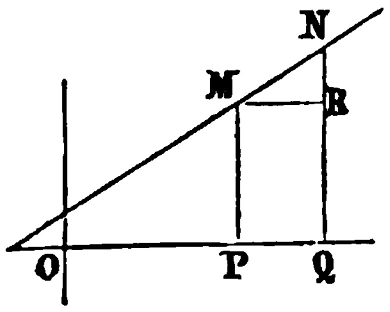
22. The equation of a straight line is,

$$y = ax + b.$$

If we take any point, as M, whose co-ordinates are y and x, and a second point N, whose co-ordinates are y', $x + h$, and we have,

$$y' - y = ah; \text{ or, } \frac{y' - y}{h} = a \quad . \quad . \quad (1.)$$

that is,



$$\frac{NR}{MR} = \text{tangent } NMR = a;$$

and, passing to the consecutive values,

$$\frac{dy}{dx} = \text{tangent } \alpha = a \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

The differential coefficient $\frac{dy}{dx}$, measures the *rate of increase* of the ordinate y, when x receives the increment dx; and since this value is independent of x, the rate will be the same for every point of the line; that is, the *rate of ascension* of the line from the axis of abscissas, is the same at every point. And since,

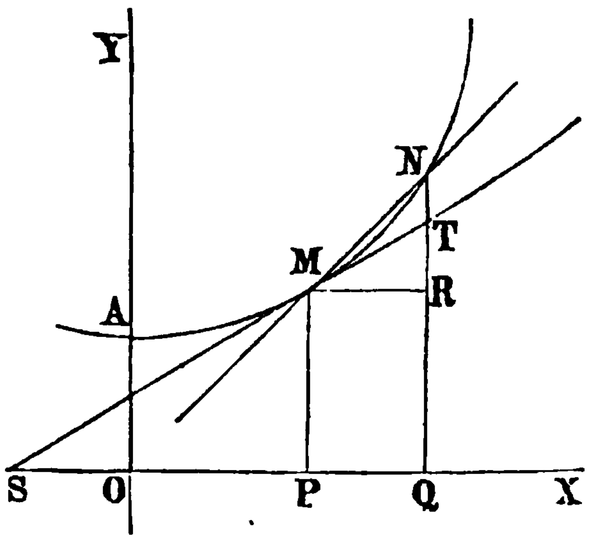
$$\frac{dy}{dx}dx = dy = adx,$$

the *change* in the value of the ordinate will be *uniform*, for uniform changes in the abscissa.

23. Let us examine an equation,

$$y = f(x) \quad . \quad . \quad (1.)$$

not of the first degree.



Let us suppose the curve AMN to be such that the abscissas and ordinates of its different points shall correspond to all possible relations between y and x, in Equation (1).

We have seen (Art. **13**) that,

$$\frac{dy}{dx} = \tan TMR = \tan \alpha; \text{ hence,}$$

the *rate of increase* of the function, or the *ascension* of the curve at any point, is equal to the tangent of the angle which the tangent line makes with the axis of abscissas. We also see, that this value of the tangent of a, will vary with the position of the point M; hence it is a function of x; therefore,

In every equation, not of the first degree, the differential coefficient is a function of the independent variable.

1. We have seen, that when the points M and N are consecutive, the secant line, MN, becomes the tangent line, TMS (Art. **13**). The line MR is then denoted by dx, and RN or RT, (for the points N and T then coincide), by dy. If we give to the new abscissa, $x + dx$, an additional increment dx, and suppose the corresponding ordinate, $y + dy$, to receive the *same increment as before*, viz.: dy, the extremity of the last ordinate will not fall on the curve, but on the tangent line, since the triangles thus formed are similar; hence,

If a function be supposed to increase uniformly from any assumed value, the differential coefficient will be constant, and equal to any increment of the function divided by the corresponding increment of the variable.

Nature of the Differential Calculus.

24. In every operation of the Differential Calculus, one of two things is always proposed, and sometimes both:

1st. To find the *rate of change* in any variable function when it *begins* to change from any assigned value.

2d. To find the difference between any two consecutive values of the function. This difference is the *actual change* in the function, produced by the smallest change which takes place in the independent variable.

The use of the independent variable is to furnish a unit of measure for the increment of the function, and thus to determine its *rate of change*, as it passes through all its states of value. This ratio can generally be expressed in numbers, either exactly or approximatively.

25. The increment of the function, corresponding to the smallest increment of the variable, being the difference between any two of its consecutive values, is a quantity of the *same kind* as the function, and differs from it only in this: that it is *too small to be expressed by numbers.* The differential of a quantity, therefore, is merely an *element* of that quantity; that is, it is the change which takes place when the quantity *begins* to increase or decrease, from any assumed value. When we find this element, we have the differential of the function; and by dividing by dx, we have the differential coefficient. Hence,

The DIFFERENTIAL CALCULUS is that branch of Mathematics which has for its object:

1. *To find the rate of change in a function, when it passes from one state of value to another, consecutive with it.*

2. *To find the actual change in the function.*

The rate of change is the *differential coefficient*, and the actual change, the *differential.*

SECTION II.

DIFFERENTIALS OF ALGEBRAIC FUNCTIONS.

Differential of sum or difference of Functions.

26. LET u be a function of the algebraic sum of several variable quantities, of the form,

$$u = y + z - w = f(x),$$

in which y, z, and w, are functions of the independent variable x.

If we give to x an increment h, we shall have,

$$u' - u = (y' - y) + (z' - z) - (w' - w);$$

hence (Art. **14**),

$$u' - u = (Ph + P'h^2) + (Qh + Q'h^2) - (Lh + L'h^2),$$

or, $$\frac{u' - u}{h} = (P + P'h) + (Q + Q'h) - (L + L'h),$$

and by passing to consecutive values,

$$\frac{du}{dx} = P + Q - L;$$

multiplying both members by dx, we have,

$$\frac{du}{dx}dx = Pdx + Qdx - Ldx.$$

But as P, Q, and L, are the differential coefficients

of y, z, and w, each regarded as a function of x, it follows (Art. 15) that,

The differential of the sum or difference of any number of functions, dependent on the same variable, is equal to the sum or difference of their differentials taken separately.

Differential of a product.

27. Let u and v denote any two functions, x the independent variable, and h its increment; we shall then have,

$$u' = u + Ph + P'h^2, \quad \text{and}$$
$$v' = v + Qh + Q'h^2,$$

and, multiplying,

$$\begin{aligned} u'v' &= (u + Ph + P'h^2)(v + Qh + Q'h^2) \\ &= uv + vPh + uQh + PQh^2 + \text{\&c.}; \end{aligned}$$

hence,

$$\frac{u'v' - uv}{h} = vP + uQ + \text{terms containing } h,\ h^2, \text{ and } h^3.$$

If now we pass to consecutive values, we have,

$$\frac{d(uv)}{dx} = vP + uQ;$$

therefore, $d(uv) = vPdx + uQdx = vdu + udv$; hence,

The differential of the product of two functions dependent on the same variable, is equal to the sum of the products obtained by multiplying each by the differential of the other.

1. If we divide by uv, we have,

$$\frac{d(uv)}{uv} = \frac{du}{u} + \frac{dv}{v} \quad . \quad . \quad . \quad . \quad . \qquad (1.)$$

that is,

The differential of the product of two functions, divided by the product, is equal to the sum of the quotients which are obtained by dividing the differential of each by its function.

28. We can easily determine, from the last formula, the differential of the product of any number of functions.

For, put $v = ts$, then,

$$\frac{dv}{v} = \frac{d(ts)}{ts} = \frac{dt}{t} + \frac{ds}{s} \quad . \quad . \quad . \quad . \qquad (2.)$$

and by substituting ts for v, in Equation (1), we have,

$$\frac{d(uts)}{uts} = \frac{du}{u} + \frac{dt}{t} + \frac{ds}{s};$$

and in a similar manner we should find,

$$\frac{d(utsr \ldots .)}{utsr \ldots .} = \frac{du}{u} + \frac{dt}{t} + \frac{ds}{s} + \frac{dr}{r} \ldots . \text{ \&c.}$$

If, in the equation,

$$\frac{d(uts)}{uts} = \frac{du}{u} + \frac{dt}{t} + \frac{ds}{s},$$

we multiply by the denominator of the first member, we shall have,

$$d(uts) = tsdu + usdt + utds; \quad \text{hence,}$$

The differential of the product of any number of functions, is equal to the sum of the products which arise

by multiplying the differential of each function by the product of all the others.

Differentials of Fractions.

29. To obtain the differential of any fraction of the form, $\frac{u}{v}$.

Put, $\frac{u}{v} = t$, then, $u = tv$.

Differentiating both members, we have,

$$du = vdt + tdv;$$

finding the value of dt, and substituting for t its value $\frac{u}{v}$, we obtain,

$$dt = \frac{du}{v} - \frac{udv}{v^2},$$

or, by reducing to a common denominator,

$$dt = \frac{vdu - udv}{v^2}; \text{ hence,}$$

The differential of a fraction is equal to the denominator into the differential of the numerator, minus the numerator into the differential of the denominator, divided by the square of the denominator.

1. If the denominator is constant, $dv = 0$, and we have,

$$dt = \frac{vdu}{v^2} = \frac{du}{v}.$$

2. If the numerator is constant, $du = 0$, and we have,

$$dt = -\frac{udv}{v^2};$$

and under this supposition, t is a decreasing function of v (Art. 7); hence, its differential coefficient should be negative (Art. 19).

Differentials of Powers.

30. To find the differential of any power of a function. First, take any function u^n, in which n is a positive whole number. This function may be considered as composed of n factors, each equal to u. Hence (Art. 27),

$$\frac{d(u^n)}{u^n} = \frac{d(uuuu\ldots.)}{(uuuu\ldots.)} = \frac{du}{u} + \frac{du}{u} + \frac{du}{u} + \frac{du}{u} + \ldots..$$

But as there are n equal factors in the numerator of the first member, there will be n equal terms in the second;

hence, $$\frac{d(u^n)}{u^n} = \frac{ndu}{u};$$

therefore, $$d(u^n) = nu^{n-1}du.$$

1. If n is fractional, denote it by $\frac{r}{s}$, and make,

$$v = u^{\frac{r}{s}}, \qquad \text{whence,} \qquad v^s = u^r;$$

and since r and s are entire numbers, we shall have,

$$sv^{s-1}dv = ru^{r-1}du;$$

from which we find,

$$dv = \frac{ru^{r-1}}{sv^{s-1}}du = \frac{ru^{r-1}}{su^{\frac{r}{s}(s-1)}}du;$$

or, by reducing,

$$dv = \frac{r}{s} u^{\frac{r}{s} - 1} du;$$

which is obtained directly from the function,

$$d(u^n) = nu^{n-1} du,$$

by changing the exponent n to $\frac{r}{s}$.

2. If the fractional exponent is one-half, the function becomes a radical of the second degree. We will give a specific rule for this class of functions.

Let $\qquad v = u^{\frac{1}{2}}, \qquad$ or, $\qquad v = \sqrt{u};$

then, $\qquad dv = \frac{1}{2} u^{\frac{1}{2} - 1} du = \frac{1}{2} u^{-\frac{1}{2}} du = \frac{du}{2\sqrt{u}};$

that is,

The differential of a radical of the second degree, is equal to the differential of the quantity under the sign divided by twice the radical.

3. Finally, if n is negative, we shall have,

$$u^{-n} = \frac{1}{u^n},$$

from which we have (Art. **29**),

$$d(u^{-n}) = d\left(\frac{1}{u^n}\right) = \frac{-d(u^n)}{u^{2n}} = \frac{-nu^{n-1} du}{u^{2n}};$$

and, by reducing,

$$d(u^{-n}) = -nu^{-n-1} du; \text{ hence,}$$

The differential of any power of a function, is equal to the exponent multiplied by the function raised to a power less one, multiplied by the differential of the function.

Formulas for differentiating Algebraic Functions.

1. $d(a) = 0$ . . . . (Art. **17.**)
2. $d(ax) = adx$ . . . . (Art. **18.**)
3. $d(x + y) = dx + dy$ . . . . . . (Art. **26.**)
4. $d(x - y) = dx - dy$ . . . . . . (Art. **26.**)
5. $d(xy) = xdy + ydx$ . . . . . (Art. **27.**)
6. $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$ . . . . . (Art. **29.**)
7. $d(x^m) = mx^{m-1}dx$ . . . . (Art. **30.**)
8. $d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}$ . . . . (Art. **30**—2.)
9. $d(x^{-\frac{r}{s}}) = -\frac{r}{s}x^{-\frac{r}{s}-1}dx$ . . . (Art. **30**—3.)

EXAMPLES.

Find the differentials of the following functions:

1. $u = ax - y$. $\quad du = adx - dy$.
2. $u = a^2x^2 + z$. $\quad du = a^2 2xdx + dz$.
3. $u = bx^2 - y^3 + a$. $\quad du = 2bxdx - 3y^2dy$.
4. $u = ax^2 - bx^3 + x$. $\quad du = (2ax - 3bx^2 + 1)dx$.
5. $u = cy^2 - x^2 + ay^2$. $\quad du = 2[(c + a)ydy - xdx.]$

6. $u = xyz.$ $\quad du = yzdx + xzdy + xydz.$

7. $u = y^2 - a^2 - 8az^5.$ $\quad du = 2(ydy - 20az^4dz.)$

8. $u = 3a^5x^n.$ $\quad du = 3na^5x^{n-1}dx.$

9. $u = -2ax^{-5} - 5 + 4b^2x^3.$ $\quad du = 2\left(\frac{5a}{x^6} + 6b^2x^2\right)dx.$

10. $u = 5x^5 - 2ay - b^2.$ $\quad du = 25x^4dx - 2ady.$

11. $u = x^n - x^3 + 46.$ $\quad du = (nx^{n-1} - 3x^2)dx.$

12. $u = ax(x^2 + 3b).$ $\quad du = 3a(x^2 + b)dx.$

13. $u = (x^2 + a)(x - a).$ $\quad du = (3x^2 - 2ax + a)dx.$

14. $u = x^2y^2z^3.$ $\quad du = 2xy^2z^3dx + 2x^2z^3ydy + 3x^2y^2z^2dz.$

15. $u = ax^2(x^3 + a).$ $\quad du = ax(5x^3 + 2a).$

16. $u = \frac{x}{y}.$ $\quad du = \frac{ydx - xdy}{y^2}.$

17. $u = \frac{a}{b - 2y^2}.$ $\quad du = \frac{4aydy}{(b - 2y^2)^2}.$

18. $u = \frac{1}{x}.$ $\quad du = \frac{-dx}{x^2}.$

19. $u = x^{-n} = \frac{1}{x^n}.$ $\quad du = \frac{-ndx}{x^{n+1}}.$

20. Find the differential of u in the equation,

$$u = \sqrt{a^2 - x^2}.$$

Put, $a^2 - x^2 = y$; then, $u = \sqrt{y^2}$; and (Art. 30—2),

$$du = \frac{dy}{2\sqrt{y}}.$$

But, $dy = -2xdx$; then, substituting for y and dy, their values, we have,

$$du = \frac{-2xdx}{2\sqrt{a^2 - x^2}} = \frac{-xdx}{\sqrt{a^2 - x^2}}.$$

21. $u = \sqrt{2ax + x^2}$ $\qquad du = \dfrac{(a + x)dx}{\sqrt{2ax + x^2}}.$

22. $u = \dfrac{1}{\sqrt{1 - x^2}}.$ $\qquad du = \dfrac{xdx}{(1 - x^2)^{\frac{3}{2}}}.$

23. $u = \dfrac{x}{x + \sqrt{1 - x^2}}.$ $\qquad du = \dfrac{dx}{\sqrt{1-x^2}\left(x+\sqrt{1-x^2}\right)^2}.$

24. $u = (a + \sqrt{x})^3.$ $\qquad du = \dfrac{3\left(a+\sqrt{x^2}\right)dx}{2\sqrt{x}}.$

25. $u = \dfrac{a^2 - x^2}{a^4 + a^2x^2 + x^4}.$

$$du = \frac{(a^4+a^2x^2+x^4)d(a^2-x^2) - (a^2-x^2)d(a^4+a^2x^2+x^4)}{(a^4+a^2x^2+x^4)^2},$$

or, $$du = \frac{-2x(2a^4 + 2a^2x^2 - x^4)dx}{(a^4 + a^2x^2 + x^4)^2}.$$

26. $u = \sqrt{a^2 + x^2} \times \sqrt{b^2 + y^2}.$

$$du = \frac{(b^2 + y^2)xdx + (a^2 + x^2)ydy}{\sqrt{a^2 + x^2}\sqrt{b^2 + y^2}}.$$

27. $u = \dfrac{x^n}{(1 + x)^n}.$ $\qquad du = \dfrac{nx^{n-1}dx}{(1 + x)^{n+1}}.$

28. $u = \dfrac{1 + x^2}{1 - x^2}.$ $\qquad du = \dfrac{4xdx}{(1 - x^2)^2}.$

29. $u = \frac{x+y}{z^3}$. $du = \frac{z(dx + dy) - (x+y)3dz}{z^4}$.

30. $u = \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}$.

$$du = -\frac{(1 + \sqrt{1-x^2})dx}{x^2\sqrt{1-x^2}}.$$

Differential of a particular binomial.

30.—1. Let $u = (a + bx^n)^m$.

Put $a + bx^n = y$; then, $u = y^m$; and (Art. **30**),

$$du = my^{m-1}dy.$$

But, from the first equation,

$$dy = nbx^{n-1}dx;$$

substituting for y and dy their values, we have,

$$du = mnb(a + bx^n)^{m-1}x^{n-1}dx;$$

that is, to find the differential of a binominal function of this form,

Multiply the exponent of the parenthesis, into the exponent of the variable within the parenthesis, into the coefficient of the variable, into the binomial raised to a power less 1, *into the variable within the parenthesis raised to a power less* 1, *into the differential of the variable.*

Rate of change of the Function.

31. What is the rate of change in the area of a square, when the side is denoted by the independent variable?

We have seen (Art. **21**) that the differential coefficient, $\frac{dy}{dx}$, denotes the *rate* of change in the function y, cor-

responding to the change dx in the value of x; and that in all equations, except those of the first degree, this rate will be *variable*, and a function of x (Art. **22**).

Let x denote the side of a square, and u its area; then,

$$u = x^2, \quad \text{and} \quad \frac{du}{dx} = 2x;$$

hence, *the rate of change in the area of a square is equal to twice its side;* that is, if the side of a square is 1, the rate of change in the area will be 2; if 5, the rate of change will be 10; and similarly for other numbers.

2. What is the rate of change in the volume of a cube, when its edge is the independent variable?

Let x denote the edge of a cube, and u its volume; then,

$$u = x^3, \quad \text{and} \quad \frac{du}{dx} = 3x^2;$$

hence, the *rate* of change in the volume, *is three times the square of its edge.* If the edge is 1, the rate of change in the volume is 3; if 2, the rate of change is 12; if 3, the rate is 27; and similarly, when the edge is denoted by other numbers.

Find the rates of change in the following functions:

3. $u = 8x^4 - 3x^2 - 5x + a.$ *A.* $32x^3 - 6x - 5.$

What will express the rate for

$$x = 1, \quad x = 2, \quad x = 3?$$

4. $u = (x^3 + a)(3x^2 + b).$ *A.* $15x^4 + 3x^2b + 6ax.$

Find the rate for,

$$x = 1, \quad x = 2.$$

5. $u = \frac{1}{1-x}$. *A.* $-\frac{1}{(1-x)^2}$.

What is the rate for, $x = 0$, $x = 4$, $x = -1$?

6. $u = (ax + x^2)^2$. *A.* $2(ax + x^2)(a + 2x)$.

What is the rate for, $x = 0$, $x = 1$, $x = 3$?

7. $u = \frac{x}{x + \sqrt{1-x^2}}$. *A.* $\frac{1}{\sqrt{1+x^2}(1+\sqrt{1-x^2})^2}$.

What is the rate for, $x = 0$, $x = 1$?

Hence, to find the rate of change for a given value of the variable: *Find the differential coefficient, and substitute the value of the variable in the second member of the equation.*

Partial Differentials.

32. If we have a function of the form,

$$u = f(x, y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

the equation denotes that u is a function of the two variables, x and y. If we suppose either of these, as y, to remain constant, and x to vary, we shall have,

$$\frac{du}{dx} = f'(x, y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

if we suppose x to remain constant, and y to vary, we shall have,

$$\frac{du}{dy} = f''(x, y) \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

The differential coefficients which are obtained under these suppositions, are called *partial differential coefficients.*

The first is the partial differential coefficient with respect to x, and the second with respect to y.

33. If we multiply both members of Equation (2) by dx, and both members of Equation (3) by dy, we obtain,

$$\frac{du}{dx}dx = f'(x, y)dx, \qquad \text{and} \qquad \frac{du}{dy}dy = f''(x, y)dy.$$

The expressions,

$$\frac{du}{dx}dx, \qquad \frac{du}{dy}dy,$$

are called, *partial differentials;* the first a partial differential with respect to x, and the second a partial differential with respect to y; hence,

A PARTIAL DIFFERENTIAL COEFFICIENT *is the differential coefficient of a function of two or more variables, under the supposition that only one of them has changed its value;* and,

A PARTIAL DIFFERENTIAL *is the differential of a function of two or more variables, under the supposition that only one of them has changed its value.*

If we suppose both the variables to undergo a change at the same time, the corresponding change which takes place in u, is called, the *total differential.* If we extend this definition to any number of variables, and assume what may be rigorously proved, viz.:

That the total differential of a function of any number of variables is equal to the sum of the partial differentials,

we have a general formula applicable to every function of two or more variables.

EXAMPLES.

1. Let $$u = x^2 + y^3 - z; \text{ then,}$$

$$\frac{du}{dx}dx = 2xdx, \quad \text{1st partial differential;}$$

$$\frac{du}{dy}dy = 3y^2dy, \quad \text{2d} \quad \text{"} \quad \text{"}$$

$$\frac{du}{dz}dz = -dz, \quad \text{3d} \quad \text{"} \quad \text{"}$$

hence, $$du = 2xdx + 3y^2dy - dz.$$

2. Let $$u = xy; \text{ then,}$$

$$\frac{du}{dx}dx = ydx,$$

$$\frac{du}{dy}dy = xdy;$$

hence, $$du = ydx + xdy.$$

3. Let $$u = x^m y^n; \text{ then,}$$

$$\frac{du}{dx}dx = mx^{m-1}y^n dx,$$

$$\frac{du}{dy}dy = ny^{n-1}x^m dy; \text{ hence,}$$

$$du = mx^{m-1}y^n dx + ny^{n-1}x^m dy = x^{m-1}y^{n-1}(mydx + nxdy).$$

4. Let $$u = \frac{x}{y}; \text{ then,}$$

$$\frac{du}{dx}dx = \frac{dx}{y},$$

$$\frac{du}{dy}dy = -\frac{xdy}{y^2};$$

hence, $$du = \frac{ydx - xdy}{y^2}.$$

5. Let $$u = \frac{ay}{\sqrt{x^2 + y^2}} = ay(x^2 + y^2)^{-\frac{1}{2}}; \text{ then,}$$

$$\frac{du}{dx}dx = -\frac{ayxdx}{(x^2 + y^2)^{\frac{3}{2}}},$$

$$\frac{du}{dy}dy = \frac{ady}{(x^2 + y^2)^{\frac{1}{2}}} - \frac{ay^2dy}{(x^2 + y^2)^{\frac{3}{2}}};$$

hence, $$du = -\frac{ayxdx - ax^2dy}{(x^2 - y^2)^{\frac{3}{2}}}.$$

6. Let $$u = xyzt; \text{ then,}$$

$$du = yztdx + xztdy + xytdz + xyzdt.$$

SECTION III.

INTEGRATION AND APPLICATIONS.

34. An Integral is a functional expression, either algebraic or transcendental, derived from a differential.

Differentiation and Integration are terms denoting operations the exact converse of each other.

Differentiation is the operation of finding the differential function from the primitive function.

Integration is the operation of finding the primitive function from the differential function.

Rules have been found for the differentiation of every form which a function can assume. Hence, in the Differential Calculus, no case can occur to which a known rule is not applicable. In the Integral Calculus it is quite otherwise.

In returning from a known differential to the integral from which it may have been derived, we *compare the differential expression with other expressions which are known to be differentials of given functions*, and thus arrive at the form of the integral, or primitive function. The main operations, therefore, of the Integral Calculus, consist in *transforming given differential expressions into others which are equivalent to, them*, and which are differentials of known functions; and thus deducing formulas applicable to all similar forms.

The integration is indicated by placing the sign $\int$

before the expression to be integrated. It is equivalent to "integral of"; thus,

$$\int 2xdx = x^2,$$

is read: "Integral of $2xdx$, is equal to x^2."

Integration of Monomials.

35. The differential of every expression of the form,

$$u = x^m, \quad \text{is} \quad du = mx^{m-1}dx \quad \text{(Art. **30**)},$$

which has been found by *multiplying the exponent into the variable raised to a power less one, into the differential of the variable.*

If, then, we have a differential expression, of the form,

$$mx^{m-1}dx, \quad \text{or,} \quad x^m dx,$$

we can find its integral by reversing the above rule; that is, to find the integral of such an expression,

Add 1 *to the exponent of the variable, and then divide by the new exponent into the differential of the variable.**

EXAMPLES.

Find the integrals of the following differential expressions:

1. If $du = 2xdx$, $\qquad \int du = \dfrac{2x^2dx}{2 \times dx} = x^2.$

2. If $du = 3x^2dx$, $\qquad \int du = \dfrac{3x^3dx}{3 \times dx} = x^3.$

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* This rule applies to every case of a differential binomial of the form, $Ax^m dx$, except that in which m is -1 (Art. **90**).

3. If $du = x^5dx$, $\int du = \frac{x^6 dx}{6 \times dx} = \frac{1}{6}x^6$.

4. If $du = x^{-3}dx$, $\int du = \frac{x^{-3+1}dx}{-2dx} = -\frac{1}{2x^2}$.

5. If $du = x^3\sqrt{x}\,dx$, $\int du = \int x^{\frac{7}{2}}dx = \frac{2}{9}x^4\sqrt{x}$.

36. We have seen, that the differential of the product of a constant by a variable, is equal to the constant multiplied by the differential of the variable (Art. **18**). Hence, *the integral of the product of a constant by a differential, is equal to the constant multiplied by the integral of the differential;* that is,

$$\int ax^m dx = a\int x^m dx = a\frac{1}{m+1}x^{m+1}.$$

Hence, *if the expression to be integrated has one or more constant factors, they should, at once, be placed as factors, without the sign of the integral.*

37. It has been shown that the differential of the sum or difference of any number of variables is equal to the sum or difference of their differentials (Art. **26**). Hence, if we have a differential expression of the form,

$$du = 2ax^2dx - bydy - z^2dz;$$

we may write,

$$\int du = 2a\int x^2dx - b\int ydy - \int z^2dz; \text{ or,}$$

$$\int du = \frac{2}{3}ax^3 - \frac{b}{2}y^2 - \frac{z^3}{3}; \text{ that is,}$$

The integral of the algebraic sum of any number of

differentials is equal to the algebraic sum of their integrals.

28. It has been shown that every constant quantity connected with a variable by the sign plus or minus, disappears in the differentiation (Art. **17**); that is,

$$d(a + x^m) = dx^m = mx^{m-1}\,dx.$$

Hence, the same differential may have several integral functions differing from each other by a constant term. Therefore, in passing from a differential to an integral expression, we must annex to the first integral obtained, a constant term, to compensate for the constant term which may have been lost in the differentiation.

For example, it has been shown in Art. (**22**), that,

$$\frac{dy}{dx} = a, \qquad \text{or,} \qquad dy = adx,$$

is the differential equation of every straight line which makes with the axis of abscissas an angle whose tangent is a. Integrating this expression, we have,

$$\int dy = a\int dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

or,

$$y = ax;$$

or, finally,

$$y = ax + C \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

If, now, the required line is to pass through the origin of co-ordinates, we shall have, for

$$x = 0, \quad y = 0, \quad \text{and consequently,} \quad C = 0.$$

But if it be required that the line shall intersect the

axis of Y at a distance from the origin equal to $+b$, we shall have, for

$x = 0, \quad y = +b, \quad$ and consequently, $\quad C = +b$;

and the true integral will be,

$$y = ax + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

If, on the contrary, it were required that the right line should intersect the axis of ordinates below the origin, we should have, for

$x = 0, \quad y = -b, \quad$ and consequently, $\quad C = -b$;

and the true integral would be,

$$y = ax - b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

The constant quantity C, which is added to the first integral, *must have such a value as to render the functional equation true for every possible value that may be attributed to the function or variable.* Hence, after having found the first integral equation, and added the constant C, *if we then make the variable equal to zero*, the value which the function assumes will be the value of C.

1. An *indefinite* integral is the first integral obtained, before the value of the constant C is determined.

2. A *particular* integral is the integral after the value of C has been found.

3. A *definite* integral is the integral corresponding to a given value of the variable.

Thus, Equation (2) is an indefinite integral, because, so long as C is undetermined, it will be the equation of a

system of parallel straight lines. Equations (3) and (4) are particular integrals, because each belongs to a particular line.

39. The zero value of an integral is called the *origin of the integral.* The value of the variable corresponding to the origin, is found by placing the second member of the integral equation equal to zero, and finding therefrom the value of the variable. Thus, if in Equation (4), we make, $y = 0$, we have,

$$ax - b = 0, \qquad \text{and} \qquad x = \frac{b}{a};$$

this shows that the origin of the value of y is on the axis of abscissas, and at a distance from the origin of coordinates equal to $\frac{b}{a}$. In Equation (3) it would be at a point whose abscissa is $-\frac{b}{a}$.

Integration between limits.

40. Having found the indefinite integral, and the particular integral, the next step is to find the definite integral; and then, the *definite* integral between given limits of the variable.

Let us take the particular integral found in Equation (3),

$$y = ax + b.$$

If it is required to find the value of the function y, for a given value of the variable x, as, $x = x'$, y will become a constant for this value, and we shall have,

$$y' = ax' + b \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

which is a *definite* integral.

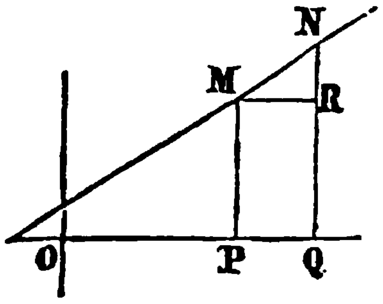
If we wish the value of the function corresponding to a second abscissa, $x = x''$, we shall have,

$$y'' = ax'' + b \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

If we subtract Equation (5) from Equation (6), we have,

$$y'' - y' = a(x'' - x') \quad . \quad . \quad . \quad . \quad (7.)$$

which is the definite integral of y, taken between the limits, $x = x'$, and $x = x''$.



If, $x' = OP$, and $x'' = OQ$; then,

$y' = PM$, and $y'' = QN$; hence,

$y'' - y' = a(x'' - x') = NR$;

Therefore: *The integral of a function, taken between two limits, is equal to the difference of the definite integrals corresponding to those limits.*

Let us now explain the *language* employed to express these relations. The modified form of Equation (1),

$$\int (dy)_{x=x'} = a\int dx,$$

is read: "Integral of y, when x is equal to x';" and

$$\int (dy)_{x=x''} = a\int dx,$$

is read: "Integral of y, when x is equal to x'';" and

$$\int_{x'}^{x''} (dy) = a\int dx,$$

is read: "Integral of the function y, taken between the limits, x' and x''; the least limit, or the limit corresponding to the subtractive integral, being placed below.

EXAMPLE.

1. What is the integral of $du = 9x^2dx$, between the limits $x = 1$, and $x = 3$, if in the primitive function u reduces to 81, when $x = 0$.

$$\int du = \int 9x^2dx = 3x^3 + C; \text{ hence,}$$

$$\int du = 3x^3 + C.$$

But from the primitive function, $u = 81$, when $x = 0$; hence, $C = 81$, and,

$$\int du = 3x^3 + 81 \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$\int (du)_{x=1} = 3 + 81 = 84 \quad . \quad . \quad (2.)$$

$$\int (du)_{x=3} = 81 + 81 = 162 \quad . \quad . \quad (3.)$$

$$\int_1^3 (du) = 162 - 84 = 78 \quad . \quad . \quad (4.)$$

What is the value of the variable corresponding to the origin of the integral (Art. **33**)?

Making the second member of Equation (1) equal 0,

$$3x^3 + 81 = 0, \qquad \text{or,} \qquad x = -3.$$

Integration of particular binomials.

41. To integrate a differential of the form (Art. **30**),

$$du = mnb(a + bx^n)^{m-1}x^{n-1}dx; \text{ or,}$$

$$du = (a + bx^n)^m x^{n-1}dx. \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

The characteristic of this form is, that *the exponent of the variable without the parenthesis is less by* 1 than the exponent of the variable within.

Put, $(a + bx^n) = z$; whence, $(a + bx^n)^m = z^m$; and,

$$nbx^{n-1}dx = dz; \quad \text{whence,} \quad x^{n-1}dx = \frac{dz}{nb}; \quad \text{hence,}$$

$$\int du = \int (a + bx^n)^m x^{n-1}dx = \int \frac{z^m dz}{nb} = \frac{z^{m+1}}{(m+1)nb};$$

and consequently,

$$u = \frac{(a + bx^n)^{m+1}}{(m+1)nb} + C.$$

Hence, to find the integral of the above form,

1. *If there is a constant factor, place it without the sign of the integral:*

2. *Augment the exponent of the parenthesis by* 1, *and then divide the quantity, with its exponent so increased, by the exponent of the parenthesis, into the exponent of the variable within the parenthesis, into the coefficient of the variable.*

EXAMPLES.

1. $$\int (a + 3x^2)^3 x dx = \frac{(a + 3x^2)^4}{4.2.3} + C; \quad \text{and}$$

2. $$\int m(a + bx^2)^{\frac{1}{2}} x dx = \frac{m}{3b}(a + bx^2)^{\frac{3}{2}} + C.$$

3. $$\int mn(a - 4cx^4)^{\frac{3}{2}} x^3 dx = -\frac{mn}{40c}(a - 4cx^4)^{\frac{5}{2}}.$$

Integration by Series.

42. The approximate integral of any function of the form,

$$du = Xdx,$$

may be found, when X is such a function of x, that it can be developed into a series. Having made the development of the function X, in the powers of x, by the Binomial Formula, we multiply each term by dx, and then integrate the terms separately. When the series is converging, we readily find the approximate value of the function for any assumed value of the variable.

EXAMPLE.

1. Find the approximate integral of,

$$\int du = \int \frac{dx}{\sqrt{1 - x^2}} = \int (1 - x^2)^{-\frac{1}{2}} dx,*$$

in which, $X = (1 - x^2)^{-\frac{1}{2}}$.

Developing, $(1 - x^2)^{-\frac{1}{2}}$, by the binomial formula,†

$$(1 - x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{3}{4}x^4 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6 + \&c.;$$

multiplying by dx, and integrating, we obtain,

$$\int du = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^7}{7} + \&c.$$

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* Bourdon, Art. **166.** University, Art. **32.**

† Bourdon, Art. **135.** University, Art. **104.**

from which we obtain an approximate value of u, corresponding to any value we may give to x.

APPLICATIONS TO GEOMETRICAL MAGNITUDES.

Equations of Tangents and Normals.

43. We have seen, that if x and y denote the co-ordinates of every point of a curve, $\frac{dy}{dx}$ will denote the tangent of the angle which the tangent line makes with the axis of abscissas (Art. **13**). This value of $\frac{dy}{dx}$ was found under the supposition that the second secant point became *consecutive with the first*; hence,

Any two consecutive points, must, at the same time, be in the chord, the curve, and the tangent.

Denote the co-ordinates of the point of tangency, in any curve, by x'' and y''. If through this point we draw any secant line, its equation will be of the form,

$$y - y'' = a(x - x'').^*$$

If the second point of secancy becomes consecutive with the first, we shall have (Art. **13**),

$$a = \frac{dy''}{dx''};$$

hence, the equation of the tangent line is,

$$y - y'' = \frac{dy''}{dx''}(x - x'') \quad . \quad . \quad . \quad . \quad (1.)$$

* Bk. I. Art. **20**.

If, in the equation of any curve, we find the value of $\frac{dy''}{dx''}$, and substitute that value in Equation (1), the equation will then denote the tangent to that curve.

1. By differentiating the equation of the circle,

$$x^2 + y^2 = R^2, \quad \text{or,} \quad x''^2 + y''^2 = R^2,$$

we have,
$$\frac{dy''}{dx''} = -\frac{x''}{y''};*$$

hence,
$$y' - y'' = -\frac{x''}{y''}(x - x'');$$

or, by reducing, $yy'' + xx'' = R^2.$

2. By differentiating the equation of the ellipse, we have,

$$\frac{dy''}{dx''} = -\frac{B^2x''}{A^2y''}.\dagger$$

3. By differentiating the equation of the parabola, we have,

$$\frac{dy''}{dx''} = \frac{p}{y''}.\ddagger$$

4. By differentiating the equation of the hyperbola, we have,

$$\frac{dy''}{dx''} = \frac{B^2x''}{A^2y''}.$$

Substituting these values, in succession, in Equation (1), and reducing, we shall find the equation of the tangent line to each curve.

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* Bk. II. Art. 8. † Bk. III. Art. 14. ‡ Bk. IV. Art. 8.

44. The equation of the normal is the form,

$$y - y'' = a'(x - x'') \quad . \quad . \quad . \quad . \quad (1.)$$

But since the normal is perpendicular to the tangent, at the point of contact,

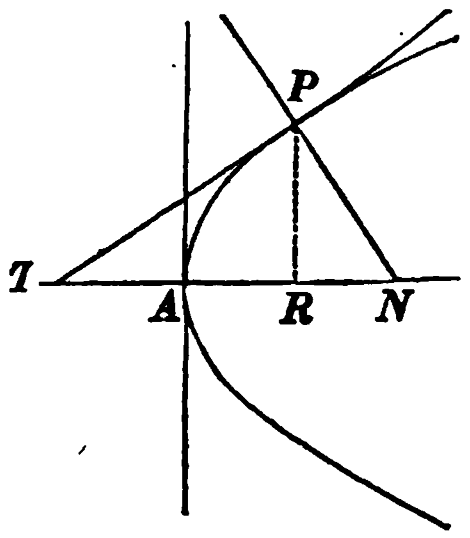
$$1 + aa' = 0;^{*} \qquad \text{or,} \qquad a' = -\frac{1}{a} = -\frac{dx''}{dy''};$$

hence, the equation of the normal is,

$$y - y'' = -\frac{dx''}{dy''}(x - x'') \quad . \quad . \quad . \quad (2.)$$

By differentiating the equation of the circle, the ellipse, the parabola, and the hyperbola, finding in each differential equation the value of $-\frac{dx''}{dy''}$, substituting that value in Equation (2), and reducing, we shall find the equation of the normal line to each curve.

Value of tangent, sub-tangent, normal, and sub-normal.



45. Let P be any point of a curve; TP the tangent, TR the sub-tangent, PN the normal, and RN the sub-normal.

Then, in the right-angled triangle TPR,

$$PR = TR \times \tan PTR = TR \times \frac{dy}{dx};$$

hence, $$TR = \frac{PR}{\frac{dy}{dx}} = y\frac{dx}{dy} = \text{Sub-tangent.}$$

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* Bk. I. Art. 23.

46. The tangent TP is equal to the square root of the sum of the squares of TR and PR; hence,

$$TP = y\sqrt{1 + \frac{dx^2}{dy^2}} = \text{Tangent.}$$

47. Since TPN is a right angle, RPN is the complement of TPR; it is therefore equal to PTR, and consequently its tangent is $\frac{dy}{dx}$; hence,

$$RN = y\frac{dy}{dx} = \text{Sub-normal.}$$

48. The normal PN is equal to the square root of the sum of the squares of PR and RN; hence,

$$PN = y\sqrt{1 + \frac{dy^2}{dx^2}} = \text{Normal.}$$

49. Apply these formulas to lines of the second order, of which the general equation is,

$$y^2 = mx + nx^2.*$$

Differentiating, we have,

$$\frac{dy}{dx} = \frac{m + 2nx}{2y} = \frac{m + 2nx}{2\sqrt{mx + nx^2}};$$

substituting this value, we find,

$$TR = y\frac{dx}{dy} = \frac{2(mx + nx^2)}{m + 2nx} = \text{Sub-tangent.}$$

$$TP = y\sqrt{1 + \frac{dx^2}{dy^2}} = \sqrt{mx + nx^2 + 4\left(\frac{mx + nx^2}{m + 2nx}\right)^2}.$$

* Bk. V. Art. **42.**

$$RN = y\frac{dy}{dx} = \frac{m + 2nx}{2} = \text{Sub-normal.}$$

$$PN = y\sqrt{1 + \frac{dy^2}{dx^2}} = \sqrt{mx + nx^2 + \frac{1}{4}(m + 2nx)^2}.$$

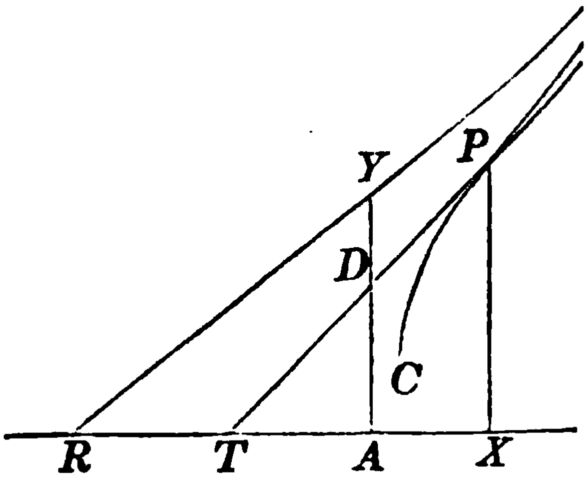
By attributing proper values to m and n, the above formulas will become applicable to each of the conic sections. In the case of the parabola, $n = 0$, and we have,

$$TR = 2x, \qquad TP = \sqrt{mx + 4x^2},$$

$$RN = \frac{m}{2}, \qquad PN = \sqrt{mx + \frac{1}{4}m^2}.$$

Asymptotes.

50. An asymptote of a curve is a line which continually approaches the curve, and becomes tangent to it at an infinite distance from the origin of co-ordinates.



Let AX and AY be the co-ordinate axes, and

$$y - y'' = \frac{dy''}{dx''}(x - x''),$$

the equation of any tangent line, as TP.

If, in the equation of the tangent, we make, in succession, $y = 0$, $x = 0$, we shall find,

$$x = AT = x'' - y''\frac{dx''}{dy''}, \qquad y = AD = y'' - x''\frac{dy''}{dx''}.$$

If the curve CPB has an asymptote RE, it is plain that the tangent PT will approach the asymptote RE, when the point of contact P, is moved along the curve from the origin of co-ordinates, and T and D will also approach the points R and Y, and will coincide with them when the co-ordinates of the point of tangency are infinite.

In order, therefore, to determine if a curve have asymptotes, we substitute in the values of AT and AD, the co-ordinates of the point which is at an infinite distance from the origin of co-ordinates. If either of the distances AT, AD, becomes finite, the curve will have an asymptote.

If both the values are finite, the asymptote will be inclined to both the co-ordinate axes; if one of the distances becomes finite and the other infinite, the asymptote will be parallel to one of the co-ordinate axes; and if they both become 0, the asymptote will pass through the origin of co-ordinates. In the last case, we shall know but one point of the asymptote, but its direction may be determined by finding the value of $\frac{dy}{dx}$, under the supposition that the co-ordinates are infinite.

51. Let us now examine the equation,

$$y^2 = mx + nx^2,$$

of lines of the second order, and see if these lines have asymptotes. We find,

$$AT = x - \frac{2y^2}{m + 2nx} = \frac{-mx}{m + 2nx},$$

$$AD = y - \frac{mx + 2nx^2}{2y} = \frac{mx}{2\sqrt{mx + nx^2}};$$

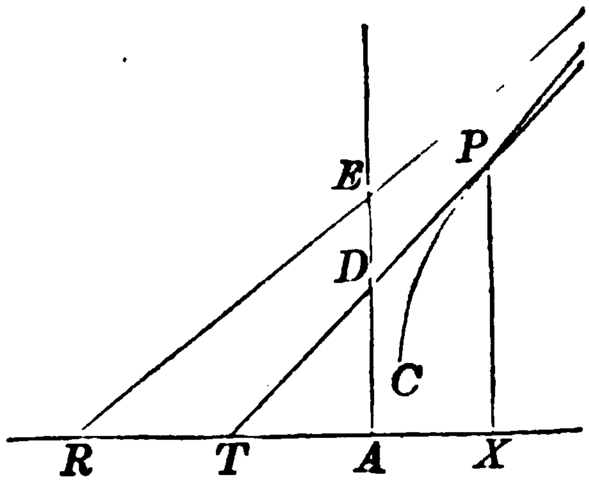
which may be put under the forms,

$$AT = \frac{-m}{\frac{m}{x} + 2n}, \qquad AD = \frac{m}{2\sqrt{\frac{m}{x} + n}},$$

and making $x = \infty$, we have,

$$AR = -\frac{m}{2n}, \quad \text{and} \quad AE = \frac{m}{2\sqrt{n}},$$

If now we make $n = 0$, the curve becomes a parabola, and both the limits, AR, AE, become infinite; hence, the parabola has no rectilinear asymptote.



If we make n negative, the curve becomes an ellipse, and AE becomes imaginary; hence, the ellipse has no asymptote.

But if we make n positive, the equation becomes that of the hyperbola, and both the values, AR, AE, become finite. If we substitute for m its value, $\frac{2B^2}{A}$, and for n its value $\frac{B^2}{A^2}$, we shall have,

$$AR = -A, \quad \text{and} \quad AE = \pm B.$$

Hence, *of the lines of the second order, the hyperbola alone has asymptotes.*

Differential of an arc.

52. We have seen that, when the points which limit any arc of a curve become consecutive, the chord, the arc, and tangent become equal (Art. **43**); therefore, *the differential of an arc is the hypothenuse* of a right-angled triangle of which the base is dx, and the perpendicular dy. Hence, if we denote any arc, referred to rectangular co-ordinates, by z, we have,

$$dz = \sqrt{dx^2 + dy^2} \quad . . (1.) \qquad \text{or,} \qquad z = \int \sqrt{dx^2 + dy^2} \quad . . (2.)$$

Rectification of a plane curve.

53. The *rectification* of a curve is the operation of finding its length; and when its length can be exactly expressed in terms of a linear unit, the curve is said to be *rectifiable.* To rectify a curve, given by its equation:

Differentiate the equation of the curve and find the value of dy^2 *in terms of* x *and* dx*; or of* dx^2 *in terms of* y *and* dy*, and substitute the value so found in the differential Equation* (2). *The second member will then contain but one variable and its differential; the integral will express the length of the arc in terms of that variable.*

EXAMPLES.

1. Find the length of the arc of a circle in terms of the radius. The equation of a circle whose radius is 1, referred to rectangular axes, when the origin is at the centre, is,

$$x^2 + y^2 = 1.$$

Denoting the arc by z, we have,

$$dz = \sqrt{dx^2 + dy^2}, \quad \text{or,} \quad z = \int \sqrt{dx^2 + dy^2}.$$

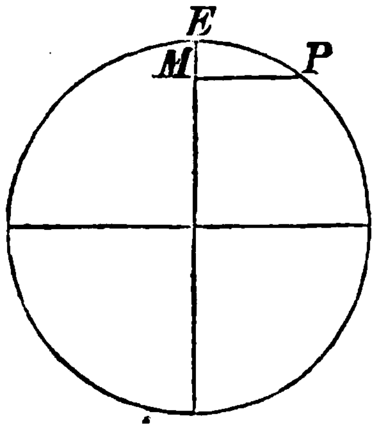
From the equation of the circle, we have,

$$xdx + ydy = 0; \quad \text{hence,} \quad dy^2 = \frac{x^2dx^2}{1 - x^2};$$

$$z = \int \sqrt{dx^2 + \frac{x^2dx^2}{1 - x^2}} = \int \frac{dx}{\sqrt{1 - x^2}} = \int (1 - x^2)^{-\frac{1}{2}}dx.$$

Developing the binomial factor into a series, by the binomial formula,* multiplying by dx, and integrating, we have (Art. **42**),

$$z = \int (1-x^2)^{-\frac{1}{2}}dx = x + \frac{1x^3}{2.3} + \frac{1.3x^5}{2.4.5} + \frac{1.3.5x^7}{2.4.6.7} + \text{\&c.} + C.$$



If we suppose the origin of the integral to be at E, the corresponding value of x will be zero, and $C = 0$. If now we integrate between the limits $x = 0$, and $x = \frac{1}{2}$, we shall obtain the value of the corresponding arc in terms of the radius 1.

But x, or PM, is the sine of the arc EP, denoted by z; and when $x = \frac{1}{2}$, $z = 30°$; hence,

$$30^0 = \int_0^{\frac{1}{2}} (1 - x^2)^{-\frac{1}{2}}dx = \frac{1}{2} + \frac{1}{2.3.2^3} + \frac{1.3}{2.4.5.2^5} + \text{\&c.},$$

* Bourdon, Art. **135.** University, Art. **104.**

hence,

$$\pi = 30° \times 6 = 6\left(\frac{1}{2} + \frac{1.1.1}{2.3.2^3} + \frac{1.3.1.1}{2.4.5.2^5} + \frac{1.3.5.1.1}{2.4.6.7.2^7} + \&c.\right),$$

and by taking the first ten terms of the series, we find,

$$\pi = 3.1415926\ldots,$$

a result true to the last decimal figure, which should be 5. We have thus found the semi-circumference of a circle whose radius is 1, or the circumference of a circle whose diameter is 1.

2. Find the length of the arc of a parabola, whose equation is,

$$y^2 = 2px.$$

Differentiating and dividing by 2, we have,

$$ydy = pdx,$$

and consequently,

$$dx^2 = \frac{y^2}{p^2}dy^2;$$

substituting this value in the differential of the arc, we have,

$$dz = \sqrt{dy^2 + \frac{y^2}{p^2}dy^2}$$

$$= \frac{1}{p}dy\sqrt{p^2 + y^2};$$

developing the radical quantity by the binomial formula, and integrating the terms separately, we have,

$$z = \left(y + \frac{1}{2}\cdot\frac{1}{3}\frac{y^3}{p^2} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{5}\frac{y^5}{p^4} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{7}\frac{y^7}{p^6} - \&c.\right) + C.$$

If we estimate the arc from the principal vertex, z and y will be zero together, and C will be zero. If we make $y = p$, z will denote the length of the arc from the vertex to the extremity of the ordinate passing through the focus.

QUADRATURES.

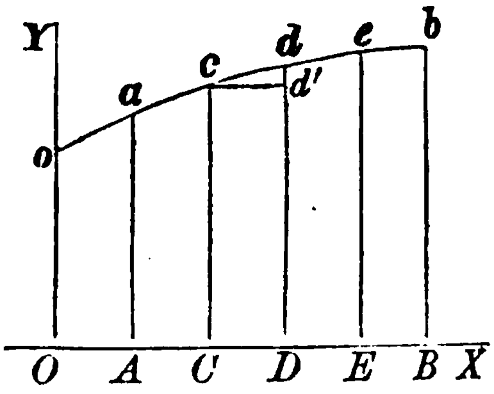
54. QUADRATURE is the operation of finding the area or measure of a surface. When this measure can be found in exact terms of the unit of measure, the surface is said to be *quadrable*.

Quadrature of plane figures.

55. A *plane figure* is a portion of a plane, bounded by lines, either straight or curved.

Let O be the origin of a system of rectangular co-ordinates, and $oacdeb$ any line whose equation is of the form,

$$y = f(x) \quad . \quad . \quad (1.)$$



If the ordinate Oo, denoted by y, move parallel to itself, along OB as a directrix, and so change its value as always to satisfy Equation (1), it will generate the plane surface $oacdebBO$, and its upper extremity will generate the line $oacdeb$. The *element*, or *differential* of this surface will be any one of the trapezoids, as $CcdD$, when the ordinates Cc and Dd are consecutive. If we denote the surface on the left of the ordinate Cc, by s, ds will denote the area of the trapezoid. This trapezoid is composed of the rectangle Cd', and the triangle $cd'd$; that is,

$$ds = ydx + \frac{dydx}{2}.$$

But since the product ydx is an infinitely small quantity of the first order, and $dydx$ an infinitely small quantity of the second order, the latter may be omitted without error (Art. **20**); hence,

$$ds = ydx; \text{ that is,}$$

The differential of a plane surface is equal to the ordinate into the differential of the abscissa.

To apply the principle enunciated in the last equation, in finding the measure of any particular plane surface:

Find the value of y in terms of x, from the equation of the bounding line; substitute this value in the differential equation, and then integrate between the required limits of x.

Nature of the Integral.

56. To comprehend the true nature of an integral, we must examine the differential from which it was derived. The differential of a plane surface is,

$$ds = ydx.$$

If we integrate between the limits $x = 0$, and $x = OB = a$, we write,

$$\int_0^a ds = \int ydx = OoacdebB;$$

that is, the first member of the equation denotes the sum of all the infinitely small rectangles between the limits $x = 0$, and $x = a$; the second member,

$$\int ydx,$$

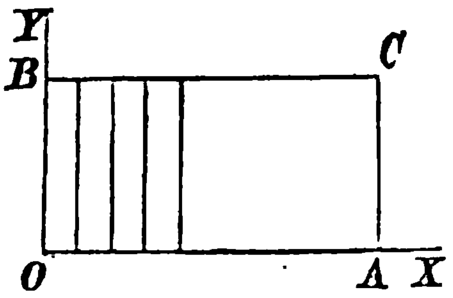
is the same thing under another form; viz.: it shows that

every value of y, between the limits $y = Oo$, and $y = Bb$, is multiplied, in succession, into each base denoted by dx; the sum of these products, each of which is ydx, is obviously the required area.

1. Perhaps the relation between the differential and the integral, may be more obvious, by observing the figure, in which the area is divided into five parts, having equal bases. If we bisect each base and draw parallel ordinates, we shall have ten parts; if we bisect again and draw parallel ordinates, we shall have twenty parts; if again, forty; and so on.

Now, there is no difficulty in seeing that each bisection doubles the number of parts, and diminishes the value of each part; and that the sum of the parts will be constantly equal to the given area. When, therefore, each part becomes *infinitely small*, any *finite* number of them is 0; but an *infinite* number is equal to a *finite* quantity, viz.: to the given area.

Area of a rectangle.



57. Let O be the origin of a system of rectangular co-ordinates. On the axis of Y, take any distance OB equal to h. Suppose the line h to move parallel to itself, along the axis of X, as a *directrix*, until it reaches the position AC. During its motion, it will generate the rectangle OC; the foot of the line will pass over every point in the line OA, and the line itself will occupy every part of the rectangle OC.

Since the equation of the line BC is,

$$y = b,$$

we shall have, for the differential of the surface,

$$ds = bdx.$$

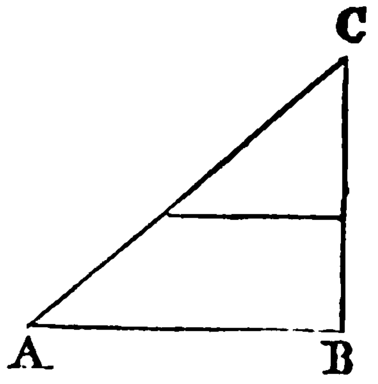
Integrating between the limits $x = 0$, and $x = b$, and observing that $C = 0$, when $x = 0$, we have,

$$\int_0^b ds = \int bdx = bx = bh; \text{ that is,}$$

The area of a rectangle is equal to the product of its base by its altitude.

Area of a triangle.

58. Let ABC be a right-angled triangle, and C the origin of co-ordinates. Denote the base AB by b, and the altitude CB by h. Denote any line parallel to the base by y, and the corresponding altitude by x.



If we suppose the base AB to be moved towards the vertex of the triangle, along CB as a directrix, and so to change its value, that,

$$b : h :: y : x, \qquad \text{or,} \qquad y = \frac{bx}{h},$$

it is plain that it will generate the surface of the triangle. If we denote the surface by s, we have,

$$ds = ydx;$$

substituting for y its value, and integrating between the limits $x = 0$, and $x = h$, we have,

$$\int_0^h ds = \frac{b}{h}\int x dx = \frac{b}{h}\frac{x^2}{2} = \frac{bh}{2};$$

that is, *The area of a triangle is equal to half the product of the base by the altitude.*

Area of the parabola.

59. Find the area of any portion of the common parabola whose equation is,

$$y^2 = 2px; \quad \text{whence,} \quad y = \sqrt{2px}.$$

This value of y being substituted in the differential equation (Art. **55**), gives (Art. **36**),

$$s = \int \sqrt{2px}dx = \sqrt{2p}\int x^{\frac{1}{2}}dx = \frac{2\sqrt{2p}}{3}x^{\frac{3}{2}} + C;$$

$$\text{or,} \quad s = \frac{2\sqrt{2px} \times x}{3} \qquad = \frac{2}{3}xy + C.$$

If we estimate the area from the principal vertex, where $x = 0$, and $y = 0$, we have, $C = 0$, and denoting the particular integral by s', we shall have,

$$s' = \frac{2}{3}xy; \quad \text{that is,}$$

The area of any portion of the parabola, estimated from the vertex, is equal to $\frac{2}{3}$ *of the rectangle of the abscissa and ordinate of the extreme point. The curve is, therefore,* QUADRABLE.

1. If it is required to find the area of the parabola from the vertex to the double ordinate through the focus, we have, for this limit, $x = \frac{1}{2}p$, $y = p$.

Denoting this integral by s'', we have,

$$\int_0^{\frac{1}{2}p} ds \;=\; s'' = \tfrac{1}{3}p^2,$$

which denotes the area bounded by the curve, the axis, and the ordinate; hence, if we double it, we shall have the required area; or,

$$2s'' = \tfrac{2}{3}p^2 = \tfrac{4}{6}p^2 = \tfrac{1}{6}(2p)^2;$$

That is, *The area is equal to one-sixth of the square described on the parameter of the axis.*

2. If the area be estimated from the ordinate through the focus, where, $x = \frac{1}{2}p$, and $y = p$, we shall have,

$$\int_0^{\frac{1}{2}p} ds \;=\; \frac{2}{3}\tfrac{1}{2}p \times p \;=\; \tfrac{1}{3}p^2 + C,$$

and since the integral is zero at its origin, we have,

$$\tfrac{1}{3}p^2 + C = 0, \qquad \text{or,} \qquad C \;=\; -\,\tfrac{1}{3}p^2.$$

Hence, the particular integral, between the limits of $x = \frac{1}{2}p$ and any value of x is,

$$\int_{\frac{1}{2}p}^{x} ds \;=\; \frac{2}{3}xy - \frac{1}{3}p^2.$$

Area of the circle.

60. The equation of the circle referred to its centre and rectangular axes is,

$$y^2 = r^2 - x^2; \qquad \text{or,} \qquad y \;=\; \sqrt{r^2 - x^2};$$

hence, the differential equation of the area (Art. 57) is,

$$ds = \left(\sqrt{r^2 - x^2}\right)dx \quad . \quad . \quad . \quad . \quad (1.)$$

in which the origin of the area is at the secondary diameter, where $x = 0$.

From Formula $\mathfrak{B}_9$ page 187, we have,

$$\int\left(\sqrt{r^2 - x^2}\right)dx = \frac{1}{2}x(r^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}r^2\int(r^2 - x^2)^{-\frac{1}{2}}dx.$$

But, by Formula (13), Art. 99, we have,

$$\int(r^2 - x)^{-\frac{1}{2}}dx = \int\frac{dx}{\sqrt{r^2 - x^2}} = \sin^{-1}\frac{x}{r} + C;$$

whence, by substitution, we have,

$$s = \frac{1}{2}x(r^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}r^2\sin^{-1}\frac{x}{r} + C \quad . \quad . \quad (2.)$$

Estimating the area from the secondary diameter, where $x = 0$, we have, $C = 0$.

If we integrate between the limits of $x = 0$, and $x = r$, we shall have one quarter of the area of the circle. When we make $x = 0$, in Equation (2), the first term in the second member becomes 0; and in the second term, $\frac{x}{r}$ becomes 1, and the arc whose sine is 1, is 90°, which is denoted by $\frac{\pi}{2}$, to the radius 1; hence,

$$\int_0^r ds = \frac{1}{2}r^2\sin^{-1}1 = \frac{1}{2}r^2 \times \frac{\pi}{2}; \text{ or,}$$

$$\text{Area of the circle} = 4\left(\frac{1}{2}r^2 \times \frac{\pi}{2}\right) = r^2\pi.$$

Area of the ellipse.

61. The equation of the ellipse, referred to its centre and axes is,

$$A^2y^2 + B^2x^2 = A^2B^2; \text{ hence,}$$

$$y = \frac{B}{A}\sqrt{A^2 - x^2},$$

and the differential equation of the area is,

$$ds = \frac{B}{A}(A^2 - x^2)^{\frac{1}{2}}dx.$$

The second member of this equation differs from the second member of Equation (1), of the last Article, only in the constant coefficient $\frac{B}{A}$, and the constant A^2 for r^2, within the parenthesis; hence, the integral of that expression becomes the integral of this, by multiplying it by $\frac{B}{A}$, and changing r into A; that is,

$$\int ds = \frac{B}{A}\left(\frac{1}{2}A^2 \times \frac{\pi}{2}\right) = \frac{AB\pi}{4}; \text{ hence,}$$

$$\text{Area of ellipse} = \frac{4AB\pi}{4} = A.B.\pi; \text{ that is,}$$

The area of an ellipse is equal to the product of its semi-axes multiplied by π.

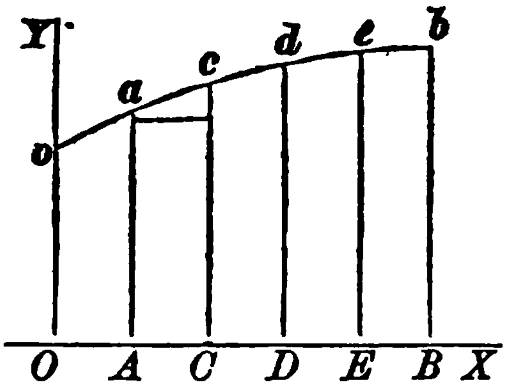
1. Let Q denote the area of a circle described on the transverse axis, and Q' the area of a circle described on the conjugate axis; then,

$$A^2\pi = Q, \qquad \text{and} \qquad B^2\pi = Q'; \text{ hence,}$$

$A^2B^2\pi^2 = QQ'$, and $AB\pi = \sqrt{Q \times Q'}$; that is,

The area of an ellipse is a mean proportional between the two circles described on its axes.

QUADRATURE OF SURFACES OF REVOLUTION.



62. Let *oacdeb* be a plane curve, *OB* the axis of abscissas, and *Oo*, *Aa*, *Cc*, &c., consecutive ordinates; then, *oa*, *ac*, *cd*, &c., will be elementary arcs. The surface described by either of these arcs, while the curve revolves around the axis *OB*, will be an element of the surface. We have seen, that when the ordinates are consecutive, the chord, the arc, and the tangent, are equal (Art. **43**); hence, the surface described by any arc, as *ac*, is equal to that described by the chord; that is, equal to the surface of the frustum of a cone, the radii of whose bases are $Aa = y$, $Cc = y + dy$, and of which the slant height $ac = \sqrt{dx^2 + dy^2}$. Hence, if we denote the surface by s, we have,*

$$ds = \pi(2y + 2y + 2dy) \times \tfrac{1}{2}\sqrt{dx^2 + dy^2};$$

or, omitting $2dy$ (Art. **20**),

$$ds = 2\pi y\sqrt{dx^2 + dy^2}; \text{ that is,}$$

The differential of a surface of revolution is equal to the circumference of a circle perpendicular to the axis, into the differential of the arc of the meridian curve.

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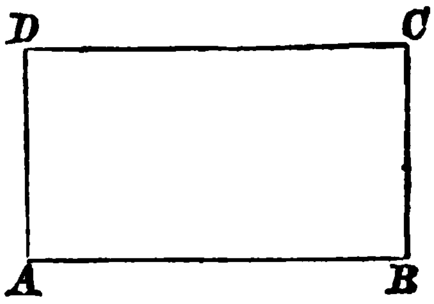
* Leg., Bk. VIII. P. 4.

Therefore, to find the measure of any surface of revolution:

Find the values of y and dy, from the equation of the meridian curve, in terms of x and dx; then substitute these values in the differential equation, and integrate between the proper limits of x.

Surface of a cylinder.

63. If the rectangle AC be revolved around the side AB, DC will generate the surface of a cylinder.



Since the generatrix is parallel to the axis AB, its equation will be,

$$y = b, \quad \text{and hence,} \quad dy = 0.$$

Substituting these values in the differential equation of the surface, we have,

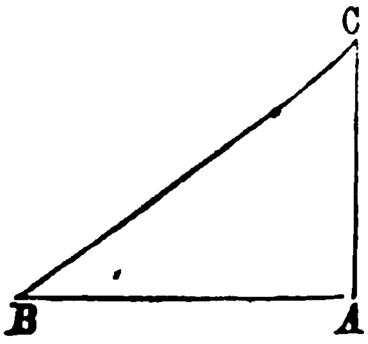
$$\int ds = \int 2\pi y \sqrt{dx^2 + dy^2} = \int 2\pi b dx = 2\pi bx + C.$$

If we suppose A to be the origin of co-ordinates, $C = 0$, and integrating between the limits $x = 0$ and $x = h$, we have,

$$s = 2b\pi h;$$

that is, *The measure of the surface of a cylinder is equal to the circumference of its base into the altitude.*

Surface of the cone.



64. If the right-angled triangle CBA be revolved around the axis AC, CB will generate the convex surface of a cone.

If we suppose C to be the origin of co-ordinates, the equation of BC will be,

$$y = ax, \quad \text{and} \quad dy = adx.$$

Substituting these values in the differential equation of the surface, we have,

$$\int ds = \int 2\pi ax\sqrt{dx^2 + a^2dx^2} = \int 2\pi axdx\sqrt{1 + a^2} + C,$$

$$\text{(Art. 35)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . = \pi ax^2\sqrt{1 + a^2} + C.$$

Estimating the surface from the vertex, where $x = 0$, we have, $C = 0$, and

$$s = \pi ax^2\sqrt{1 + a^2}.$$

If we make $x = h = AC$, and $BA = b$, we have, $a = \dfrac{b}{h}$, and consequently,

$$s = \pi\frac{b}{h}\,h^2\sqrt{1 + \frac{b^2}{h^2}} = \frac{2\pi b\sqrt{h^2 + b^2}}{2} = 2\pi b \times \frac{BC}{2}.$$

that is, *The convex surface of a cone is equal to the circumference of the base into half the slant height.*

Surface of the sphere.

65. To find the surface of a sphere. The equation of the meridian curve, referred to the centre, is,

$$x^2 + y^2 = R^2.$$

By differentiating, we have,

$$xdx + ydy = 0;$$

hence,

$$dy = -\frac{xdx}{y}, \qquad \text{and} \qquad dy^2 = \frac{x^2dx^2}{y^2}.$$

Substituting for dy^2 its value, in the differential of the surface, which is,

$$ds = 2\pi y\sqrt{dx^2 + dy^2},$$

we have,

$$\int ds = \int 2\pi y\sqrt{dx^2 + \frac{x^2}{y^2}dx^2} = \int 2\pi R dx = 2\pi R x + C.$$

If we estimate the surface from the plane passing through the centre, and perpendicular to the axis of X, we shall have,

$$s = 0, \quad \text{for} \quad x = 0, \quad \text{and consequently,} \quad C = 0.$$

To find the entire surface of the sphere, we must integrate between the limits $x = +R$, and $x = -R$, and then take the sum of the integrals, without reference to their algebraic signs; for, these signs only indicate the position of the parts of the surface with respect to the plane passing through the centre.

Integrating between the limits,

$$x = 0, \qquad \text{and} \qquad x = +R,$$

we find, $$s = 2\pi R^2;$$

and integrating between the limits $x = 0$, and $x = -R$, there results,

$$s = -2\pi R^2;$$

hence,

$$\text{Surface} = 4\pi R^2 = 2\pi R \times 2R;$$

that is, *Equal to four great circles, or equal to the curved surface of the circumscribing cylinder.*

1. The two equal integrals,

$$s = 2\pi R^2, \qquad \text{and} \qquad s = -2\pi R^2,$$

indicate that the surface is divided into two equal parts by the plane passing through the centre.

Surface of the paraboloid.

66. To find the surface of the paraboloid of revolution. Take the equation of the meridian curve,

$$y^2 = 2px,$$

which being differentiated, gives,

$$dx = \frac{ydy}{p}, \qquad \text{and} \qquad dx^2 = \frac{y^2dy^2}{p^2}.$$

Substituting this value of dx in the differential of the surface, we have,

$$ds = 2\pi y\sqrt{\left(\frac{y^2 + p^2}{p^2}\right)dy^2} = \frac{2\pi}{p}ydy\sqrt{y^2 + p^2}.$$

But we have found (Art. 41),

$$\int \frac{2\pi}{p} y dy \sqrt{y^2 + p^2} = \frac{2\pi}{3p}(y^2 + p^2)^{\frac{3}{2}} + C;$$

hence,

$$s = \frac{2\pi}{3p}(y^2 + p^2)^{\frac{3}{2}} + C.$$

If we estimate the surface from the vertex, at which point $y = 0$, we shall have,

$$0 = \frac{2\pi p^2}{3} + C, \qquad \text{whence,} \qquad C = -\frac{2\pi p^2}{3};$$

and integrating between the limits,

$$y = 0, \qquad \text{and} \qquad y = b,$$

we have,

$$s = \frac{2\pi}{3p}[(b^2 + p^2)^{\frac{3}{2}} - p^3].$$

Surface of the ellipsoid.

67. To find the surface of an ellipsoid described by revolving an ellipse about the transverse axis.

The equation of the meridian curve is,

$$A^2y^2 + B^2x^2 = A^2B^2,$$

whence,

$$dy = -\frac{B^2}{A^2}\frac{xdx}{y} = -\frac{B}{A}\frac{xdx}{\sqrt{A^2 - x^2}};$$

substituting the square of dy in the differential of the surface, and for y its value,

$$\frac{B}{A}\sqrt{A^2 - x^2},$$

we have,

$$ds = 2\pi\frac{B}{A^2}dx\sqrt{A^4 - (A^2 - B^2)x^2}; \quad (1.)$$

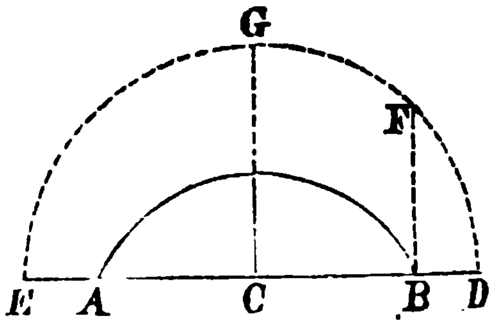
hence, $$\int ds = 2\pi\frac{B}{A^2}\sqrt{A^2 - B^2}\int dx\sqrt{\frac{A^4}{A^2 - B^2} - x^2}.$$

Put, $2\pi\frac{B}{A^2}\sqrt{A^2 - B^2} = D$, a constant quantity;

and $$\frac{A^4}{A^2 - B^2} = R^2, \text{ also a constant,}$$

and we have,

$$\int ds = D\int dx\sqrt{R^2 - x^2}.$$



With C, the centre of the meridian curve, and the radius R, describe a semi-circle. Then, $\int dx\sqrt{R^2 - x^2}$, is a circular segment of which the abscissa is x, and radius R.

If, then, we estimate the surface of the ellipsoid from the plane passing through the centre, and estimate the area of the circular segment from the same plane, any portion of the surface of the ellipsoid will be equal to the corresponding portion of the circle, multiplied by the constant D. Hence, if we integrate the expression,

$$\int dx\sqrt{R^2 - x^2},$$

between the limits $x = 0$, and $x = A$, we shall have the area of the segment $CGFB$, which denote by D'. Hence,

$$\tfrac{1}{2} \text{ surface ellipsoid } = D \times D'; \text{ and}$$

$$\text{Surface} = 2D \times D'.$$

1. If we make $A = B$, in Equation (1), the ellipsoid becomes a sphere, and we have,

$$s = \int 2\pi R dx = 2\pi Rx + C.$$

If we estimate the surface from the plane passing through the centre, $C = 0$, and integrate between the limits $x = 0$, and $x = R$, we have,

$$\tfrac{1}{2} \text{ surface of sphere } = 2\pi R^2; \text{ hence,}$$

$$\text{Surface} = 4\pi R^2.$$

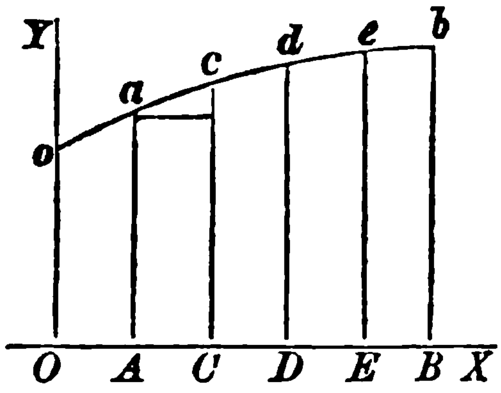
CUBATURE OF VOLUMES OF REVOLUTION.

68. Cubature is the operation of finding the measure of a volume. When this measure can be found in exact terms of the measuring cube, the volume is said to be *cubable*.

69. A *volume of revolution* is a volume generated by the revolution of a plane figure about a fixed line, called the *axis*.

If the plane figure $OoacdebB$, be revolved about the axis of X, it will generate a volume of revolution.

Let us suppose the ordinates Aa, Cc, Dd, &c., to be consecutive. During the revolution, any element of the surface, as, $AacC$, will generate the frustum of a cone, of which the radii of the bases are $Aa = y$, $Cc = y + dy$, and the altitude, $AC = dx$. This frustum will be an element of the volume, and will have for its measure,*



$$\frac{\pi}{3}[y^2 + (y + dy)^2 + y(y + dy)]dx.$$

If we denote the volume by V, develop the terms within the parenthesis, multiply by dx, and then reject all the terms containing the infinitely small quantities of the second order (Art. **20**), we shall have,

$$dV = \pi y^2 dx.$$

The area of a circle described by any ordinate y, is πy^2†; hence, *The differential of a volume of revolution is equal to the area of a circle perpendicular to the axis into the differential of the axis.*

The differential of a volume generated by the revolution of a plane figure about the axis of Y, is $\pi x^2 dy$.

70. To find the value of V for any given volume:

Find the value of y^2 in terms of x, from the equation of the meridian curve; substitute this value in the differential equation, and then integrate between the required limits of x.

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* Leg., Bk. VIII. P. 6. † Leg., Bk. V. Prop. 16.

EXAMPLES.

1. Find the volume of a right cylinder with a circular base, whose altitude is h and the radius of whose base is r.

We have for the differential of the volume,

$$dV = \pi y^2 dx;$$

and since $y = r$, we have,

$$dV = \int \pi r^2 dx;$$

integrating between the limits $x = 0$, and $x = h$,

$$\int_0^h dV = V = \pi r^2 x = \pi r^2 h; \text{ that is,}$$

*The measure of the volume of a cylinder is equal to the area of its base multiplied by the altitude.**

2. Find the volume of a right cone with a circular base, whose altitude is h, and the radius of the base, r.

If we suppose the vertex of the cone to be at the origin of co-ordinates, and the axis to coincide with the axis of abscissas, we shall have,

$$y = ax, \quad \text{or,} \quad y = \frac{r}{h}x, \quad \text{and} \quad y^2 = \frac{r^2}{h^2}x^2;$$

substituting this value of y^2, we have,

$$\int dV = \int \pi \frac{r^2}{h^2} x^2 dx.$$

---

* Legendre, Bk. VIII. Prop. 2.

Integrating between the values $x = 0$, and $x = h$,

$$\int_0^h dV = V = \pi\frac{r^2}{h^2}\frac{x^3}{3} = \pi r^2 \times \frac{h}{3}; \text{ that is,}$$

*The measure of the volume of a cone is equal to the area of the base into one-third of the altitude.**

3. To find the volume of a prolate spheroid.†

The equation of the meridian curve is,

$$A^2y^2 + B^2x^2 = A^2B^2; \quad \text{hence,} \quad y^2 = \frac{B^2}{A^2}(A^2 - x^2).$$

and
$$dV = \pi\frac{B^2}{A^2}(A^2 - x^2)dx; \text{ hence,}$$

$$V = \frac{\pi B^2}{A^2}\left(A^2x - \frac{x^3}{3}\right) + C,$$

$$= \frac{\pi B^2}{3A^2}(3A^2x - x^3) + C.$$

If we estimate the volume from the plane passing through the centre, we have, for $x = 0$, $V = 0$, and consequently, $C = 0$; and taking the integral between the limits $x = 0$, and $x = A$, we have,

$$\int_0^A dV = \frac{2}{3}\pi B^2 \times A;$$

which is half the volume; consequently, the entire volume,

$$2V = \frac{2}{3}\pi B^2 \times 2A$$

---

* Legendre, Bk. VII. Prop. 5. † Bk. VI. Art. 37.

But, πB^2 expresses the area of a circle described on the conjugate axis, and $2A$ is the transverse axis; hence,

The volume of a prolate spheroid is equal to two-thirds of the circumscribing cylinder.

1. If an ellipse be revolved around the conjugate axis, it will describe an oblate spheroid, and we shall have,

$$dV = \int \pi x^2 dy;$$

substituting for x^2, and integrating, we have,

$$2V = \frac{2}{3}\pi A^2 \times 2B;$$

that is, two-thirds of the circumscribing cylinder.

2. If we compare the two results together, we find,

oblate spheroid : prolate spheroid : : A : B.

3. If we make $A = B$, the ellipsoid becomes a sphere whose diameter is the transverse axis. Then,

$$2V = \frac{2}{3}\pi R^2 \times D = \frac{1}{6}\pi D^3;$$

that is, *Equal to two-thirds of the circumscribing cylinder, or to one-sixth of π into the cube of the diameter.*

4. Find the volume of a paraboloid. The equation of the meridian curve is,

$$y^2 = 2px; \quad \text{hence,}$$

$$dV = 2\pi px dx, \qquad \text{and} \qquad V = \pi px^2.$$

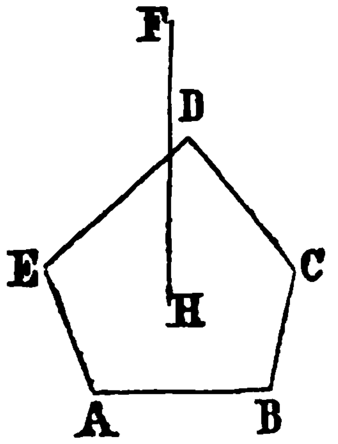
If we estimate the volume from the vertex, $C = 0$. If we integrate between the limits $x = 0$, and $x = h$, and designate by b, the ordinate corresponding to the abscissa $x = h$, we have,

$$V = \pi p h^2 = \pi b^2 \times \frac{h}{2}; \text{ that is,}$$

equal to half the cylinder having the same base and altitude.

Prism and Pyramid.

1. Let $ABCDE$ be any polygon, and FH a line perpendicular to the plane of the base. If the polygon move along the line FH, parallel to itself, it will generate a prism. If we denote the volume by V, the area of the base by b, and the indefinite line HF by x, we shall have,



$$dV = bdx.$$

and, integrating between the limits $x = 0$, and $x = h$,

$$\int_0^h dV = \int bdx = b \times x = b \times h.$$

2. If we suppose the base so to vary, as it moves along the line FH, as to bear a constant ratio to the square of its distance from the point F, it will generate the volume of a pyramid, of which F is the vertex and $ABCDE$ the base.* If we denote the variable generatrix, at any point, by y, and its distance from the vertex by x, we have,

$$dV = ydx.$$

But, $\quad b : y :: h^2 : x^2$; hence, $\quad y = \frac{b}{h^2} \times x^2$;

therefore, $\quad dV = \frac{b}{h^2} \times x^2 dx$;

and integrating between the limits $x = 0$, and $x = h$, we have,

$$\int_0^h dV = \frac{b}{h^2} \int x^2 dx = \frac{b}{h^2} \times \frac{x^3}{3} = b \times \frac{h}{3}.$$

* Legendre, Bk. VII. P. 3. Cor.

SECTION IV.

SUCCESSIVE DIFFERENTIALS — SIGNS OF DIFFERENTIAL CO-EFFICIENTS — FORMULAS OF DEVELOPMENT.

Successive Differentials.

71. If u denotes any function, and x the independent variable, we have seen that the differential coefficient P, is, in general, a function of x (Art. **23**). It may therefore be differentiated, and a new differential coefficient will thus be obtained, which is called the *second differential coefficient.*

72. In passing from the function u to the first differential coefficient, the exponent of x is diminished by 1, in every term where x enters (Art. **30**); hence, the *relation* between the primitive function u and the variable x, is different from that which exists between the first differential coefficient and x. Hence, the same change in x, will occasion *different degrees* of change in the *primitive function* and in the *first differential coefficient.*

The second differential coefficient will, in general, be a function of x, exhibiting a still different relation; hence, a new differential coefficient may be formed from it, which may also be a function of x; and so on, for succeeding differential coefficients.

If we designate the successive differential coefficients by

$$p, \quad q, \quad r, \quad s, \quad \&c.,$$

we shall have,

$$\frac{du}{dx} = p, \qquad \frac{dp}{dx} = q, \qquad \frac{dq}{dx} = r, \ \&c.; \ \text{and}$$

$$du = pdx, \qquad dp = qdx, \qquad dq = rdx.$$

But the differential of p may be obtained by differentiating its value $\frac{du}{dx}$, regarding the denominator dx as constant; we therefore have,

$$d\left(\frac{du}{dx}\right) = dp, \qquad \text{or,} \qquad \frac{d^2u}{dx} = dp;$$

substituting for dp its value, and dividing by dx,

$$\frac{d^2u}{dx^2} = q.$$

The notation, d^2u, indicates that the function u has been differentiated twice; it is read, *second differential of* u. The denominator dx^2, denotes *the square of the differential of* x, and not the differential of x^2. It is read: differential of x, *squared.*

If we differentiate the value of q, we have,

$$d\left(\frac{d^2u}{dx^2}\right) = dq, \qquad \text{or,} \qquad \frac{d^3u}{dx^2} = dq;$$

hence, $$\frac{d^3u}{dx^3} = r, \quad \&c.;$$

and in the same manner we may find,

$$\frac{d^4u}{dx^4} = s.$$

The third differential coefficient, $\frac{d^3u}{dx^3}$, is read: third differential of u, divided by dx cubed; and the differential coefficients which succeed it are read in a similar manner.

Hence, the successive differential coefficients are,

$$\frac{du}{dx} = p, \qquad \frac{d^2u}{dx^2} = q, \qquad \frac{d^3u}{dx^3} = r, \qquad \frac{d^4u}{dx^4} = s, \quad \text{\&c.},$$

from which we see, that each differential coefficient is derived from the one that immediately precedes it, in the same way as the first is derived from the primitive function.

The differentials of the different orders are obtained by multiplying the differential coefficients by the corresponding powers of dx; thus,

$$\frac{du}{dx}dx = \text{1st differential of } u,$$

$$\frac{d^2u}{dx^2}dx^2 = \text{2d differential of } u,$$

. . . .

$$\frac{d^nu}{dx^n}dx^n = n\text{th differential of } u.$$

EXAMPLES.

1. Find the differential coefficients in the function,

$$u = ax^3.$$

$$\frac{du}{dx} = 3ax^2 = p,$$

$$\frac{du^2}{dx^2} = 6ax = q,$$

$$\frac{du^3}{dx^3} = 6a = r.$$

2. Find the differential coefficients in the function,

$$u = ax^n.$$

The first differential coefficient is,

$$\frac{du}{dx} = nax^{n-1}.$$

Since n, a, and dx, are constants, we have for the second differential coefficient,

$$\frac{d^2u}{dx^2} = n(n-1)ax^{n-2};$$

and for the third,

$$\frac{d^3u}{dx^3} = n(n-1)(n-2)ax^{n-3};$$

and for the fourth,

$$\frac{d^4u}{dx^4} = n(n-1)(n-2)(n-3)ax^{n-4}.$$

It is plain, that when n is a positive integral number, the function

$$u = ax^n,$$

will have n differential coefficients. For, when n differentiations have been made, the exponent of x in the second member will be 0; hence, the nth differential coefficient will be a constant, and the succeeding ones will be 0. Thus,

$$\frac{d^n u}{dx^n} = n(n-1)(n-2)(n-3) \ldots\ldots a.1,$$

and
$$\frac{d^{n+1}u}{dx^{n+1}} = 0.$$

Sign of the first differential coefficient.

72. If we have a curve whose equation is,

$$y = f(x),$$

and give to x any increment h, we have (Art. **13**),

$$\frac{y'-y}{h} = \frac{f(x+h) - f(x)}{h},$$

and passing to the consecutive values,

$$\frac{dy}{dx} = \tan \alpha.$$

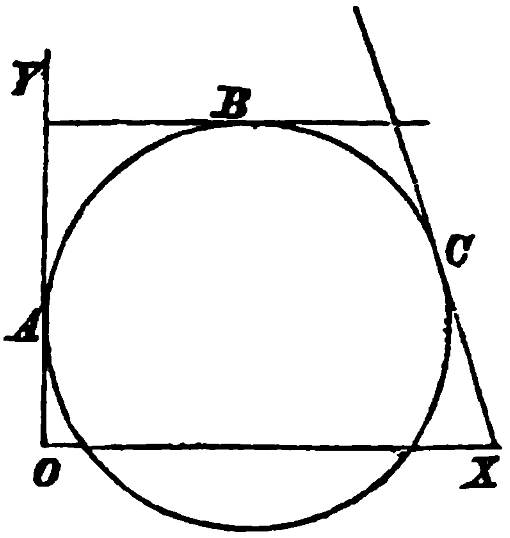
If we so place the origin of co-ordinates that the curve shall lie within the first angle, h will be positive, and $y' - y$ will be positive at all points where the curve

recedes from the axis of X, and negative where it approaches the axis; and this is true for consecutive as well as for other values. Hence, *the curve will recede from the axis of X when the first differential coefficient is positive, and approach the axis when that coefficient is negative.*

The general proposition for all the angles and every possible relation of y and x, is this:

The curve will recede from the axis of X when the ordinate and first differential coefficient have the same sign, and approach it when they have different signs.

1. To determine whether a given curve, as ABC, recedes from, or approaches to the axis of X, at any point, as C: Find, from the equation of the curve, the first differential coefficient, and see whether it is positive or negative.



2. If the tangent becomes parallel to the axis of X at any point, as B,

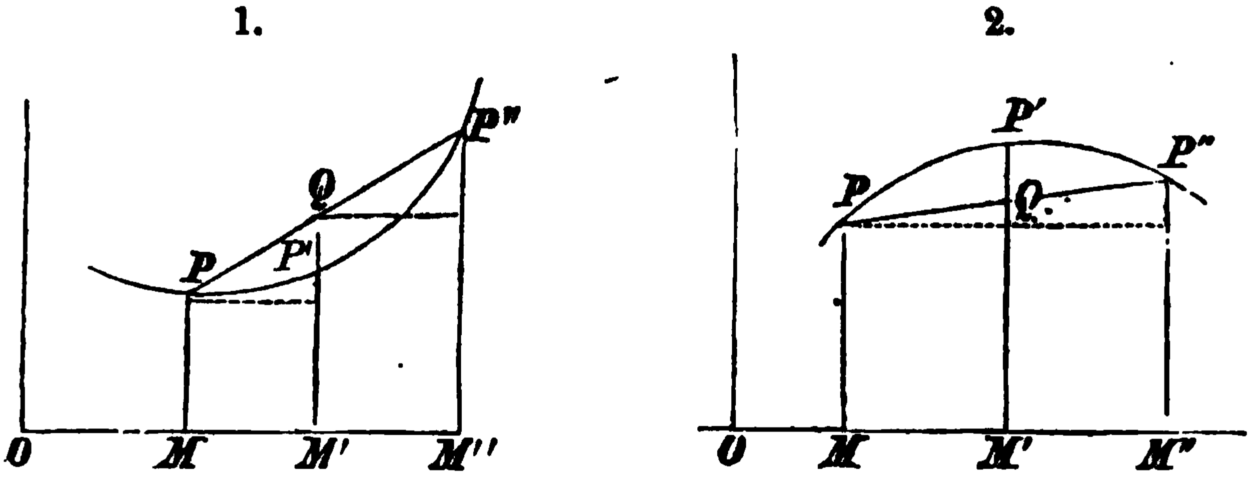
$$\frac{dy}{dx} = \tan \alpha = 0; \qquad \text{hence,} \qquad \alpha = 0.$$

If the tangent becomes perpendicular to the axis of X, at any point, as A

$$\frac{dy}{dx} = \tan \alpha = \infty; \qquad \text{hence,} \qquad \alpha = 90°.$$

Sign of the second differential coefficient.

73. A curve is *convex* towards the axis of abscissas when it lies between the chord and the axis; and *concave*, when the chord lies between the curve and the axis.



Figures (1) and (2) denote two curves, the one convex and the other concave towards the axis of X.

Let PM be any ordinate of either curve, $P'M'$ an ordinate consecutive with it, and $P''M''$ an ordinate consecutive with $P'M'$.

If we designate the ordinate PM by y, $P'Q'$ will be denoted by dy (Art. **21**), and we shall have,

$$P'M' = y + dy;$$

and since $P''M''$ is consecutive with $P'M'$,

$$\begin{aligned} P''M'' &= y + \ dy + d(y + dy) \\ &= y + 2dy + d^2y. \end{aligned}$$

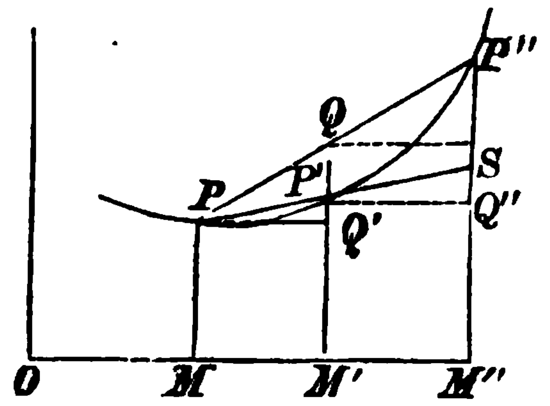
Since, $\quad MM' = M'M'' = dx, \quad QM' = \dfrac{MP + P''M''}{2};$

hence, $\quad QM' = \dfrac{y + y + 2dy + d^2y}{2} = y + dy + \dfrac{d^2y}{2},$

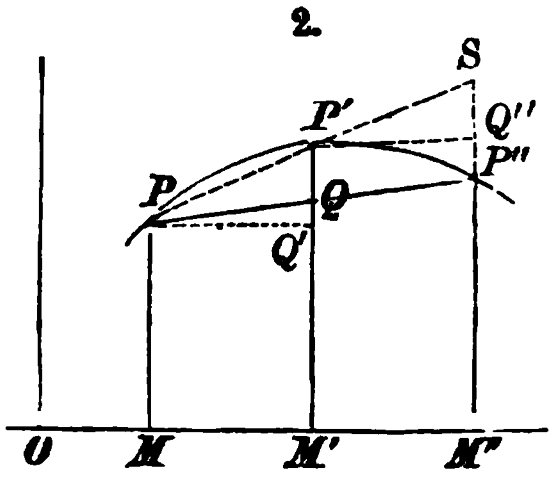
and $\quad QM' - P'M' = QP' = \dfrac{d^2y}{2}.$

In the case of *convexity*, $QM' > P'M'$, and then, d^2y is positive.

1.



2.



In the case of *concavity*, $QM' < P'M'$, and then, d^2y is negative; and since dx^2 is always positive, the second differential coefficient will have the same sign as the second differential. of y.

If we take the case in which the ordinates are negative, the second differential coefficient will still have the same sign as the ordinate, when the curve is convex, and a different sign when it is concave. Hence,

The second differential coefficient will have the same sign as the ordinate when the curve is convex towards the axis of abscissas, and a contrary sign when it is concave.

1. The second differential of y is derived from dy in the same way that dy is derived from y (Art. 72); viz.: by producing the chord PP', and finding the difference of the consecutive values of $P''Q''$ and SQ'', which is $P''S$.

The co-ordinates x and y determine a single point of the curve, as P; these, in connection with dx and dy, determine a second point, P', consecutive with the first; and these two sets of values, in connection with the second differential of y, determine a third point, P'', consecutive with P'.

Hence, the co-ordinates x and y, and the first and second differential coefficients, *always determine three consecutive points of a curve.*

2. When the curve is convex towards the axis of abscissas, the tangent of the angle which the tangent line makes with the axis of X, is an increasing function of x; hence, its differential coefficient, that is, the *second* differential of the function, ought to be, as we have found it, positive (Art. **19**).

When the curve is concave, the first differential coefficient is a decreasing function of the abscissas; hence, the second differential coefficient should be negative (Art. **19**).

Applications.

74. The equation of the circle, referred to its centre and rectangular axes, is,

$$x^2 + y^2 = R^2; \qquad \text{hence,} \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

Placing $\quad -\frac{x}{y} = 0, \quad$ we have, $\quad x = 0.$

Substituting this value of x in the equation of the circle, we have,

$$y = \pm R;$$

hence, the tangent is parallel to the axis of abscissas at the two points where the axis of ordinates intersects the circumference.

If we make, $\quad \frac{dy}{dx} = -\frac{x}{y} = \infty, \quad$ we have, $\quad y = 0;$

substituting this value in the equation of the circle,

$$x = \pm R; \quad \text{hence,}$$

the tangent is perpendicular to the axis of abscissas at the points where the axis intersects the circumference.

1. For the second differential coefficient, we find,

$$\frac{d^2y}{dx^2} = -\frac{R^2}{y^3},$$

which will be negative when y is positive, and positive when y is negative. Hence, the circumference of the circle is concave towards the axis of abscissas.

2. If we apply the same process to the equation of the ellipse, of the parabola, and of the hyperbola, we shall find that the tangents, at the principal vertices, are parallel to the axes of ordinates; that the second differential coefficient and ordinate, in all the cases, except that of the opposite hyperbolas, have contrary signs; and hence, *all the curves, except the conjugate hyperbolas, are concave towards the axis of abscissas.*

MACLAURIN'S THEOREM.

75. MACLAURIN'S THEOREM explains the method of developing into a series any function of a single variable.

Let u denote any function of x, as, for example,

$$u = (a + x)^m \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

It is required to develop this, or any other function of x, into a series of the form,

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. \quad . \quad . \quad (2.)$$

in which A, B, C, D, &c., are independent of x, and arbitrary functions of the constants which enter into the

second member of Equation (1). When these coefficients are found, the form of the series will be known.

Since the coefficients, A, B, C, &c., are, by hypothesis, independent of x, each will have the same value for $x = 0$, as for any other value of x; hence, it is only necessary to determine them for $x = 0$.

If we make $x = 0$, in Equation (2), all the terms in the second member, after the first, will become zero, and the second member will reduce to A, which is what the function u becomes in Equation (1), when $x = 0$ That value is thus indicated:

$$(u)_{x=0} = A.$$

If we find the successive differential coefficients of u, from Equation (2), we shall have,

$$\frac{du}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \text{\&c.}$$

$$\frac{d^2u}{dx^2} = 2C + 2.3Dx + 3.4Ex^2 + \text{\&c.}$$

$$\frac{d^3u}{dx^3} = 2.3D + 2.3.4x + \text{\&c.}$$

&c., &c.;

whence,

$$A = (u)_{x=0}$$

$$B = \left(\frac{du}{dx}\right)_{x=0}$$

$$C = \frac{1}{1.2}\left(\frac{d^2u}{dx^2}\right)_{x=0}$$

$$D = \frac{1}{1.2.3}\left(\frac{d^3u}{dx^3}\right)_{x=0} \qquad \text{\&c.,} \qquad \text{\&c.;}$$

hence,

$$u = (u)_{x=0} + \left(\frac{du}{dx}\right)_{x=0} x + \frac{1}{1.2}\left(\frac{d^2u}{dx^2}\right)_{x=0} x^2$$

$$+ \frac{1}{1.2.3}\left(\frac{d^3u}{dx^3}\right)_{x=0} x^3 + \&c. \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

which is Maclaurin's Formula. In applying the formula, we omit the expressions $x = 0$, although *the coefficients are always found under this hypothesis.*

EXAMPLES.

1. Develop $(a + x)^m$, by Maclaurin's Formula,

$$A = a^m,$$

$$B = \left(\frac{du}{dx}\right) = m(a + x)^{m-1} = ma^{m-1},$$

$$C = \frac{1}{2}\left(\frac{d^2u}{dx^2}\right) = \frac{m(m-1)}{1.2}(a + x)^{m-2} = \frac{m(m-1)}{1.2}a^{m-2},$$

$$D = \frac{1}{1.2.3}\left(\frac{d^3u}{dx^3}\right) = \frac{m}{1}\frac{(m-1)}{2}\frac{(m-2)}{3}(a + x)^{m-3}$$

$$= \frac{m}{1}\frac{(m-1)}{2}\frac{(m-2)}{3}a^{m-3},$$

&c., &c., &c.

Substituting these values in Equation (1), we have,

$$(a + x)^m = a^m + ma^{m-1}x + \frac{m}{1}\frac{(m-1)}{2}a^{m-2}x^2$$

$$+ \frac{m}{1}\frac{(m-1)}{2}\frac{(m-2)}{3}a^{m-3}x^3 + \&c.;$$

the same result as found by the Binomial Formula.

2. If the function is of the form,

$$u = \frac{1}{a+x} = (a+x)^{-1} = a^{-1}\left(1 + \frac{x}{a}\right)^{-1}.$$

we find,

$$A = \frac{1}{a},$$

$$B = \left(\frac{du}{dx}\right)_{x=0} = -1(a+x)^{-2} = -\frac{1}{(a+x)^2} = -\frac{1}{a^2},$$

$$C = \frac{1}{2}\left(\frac{d^2u}{dx^2}\right)_{x=0} = \frac{-1 \times -2(a+x)^{-3}}{2} = \frac{1}{a^3},$$

$$D = \frac{1}{2.3}\left(\frac{d^3u}{dx^3}\right)_{x=0} = \frac{-1 \times -2 \times -3(a+x)^{-4}}{2.3} = -\frac{1}{a^4},$$

&c., &c., &c.

Substituting these values in Maclaurin's Formula,

$$\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \frac{x^5}{a^6} - \frac{x^7}{a^8} + \&c.$$

3. Develop into a series, the function,

$$u = \sqrt{a^2 + x^2} = a\left(1 + \frac{x^2}{a^2}\right)^{\frac{1}{2}}.$$

4. Develop into a series, the function,

$$u = \sqrt[3]{(a^2 - x^2)^2} = a^{\frac{4}{3}}\left(1 - \frac{x^2}{a^2}\right)^{\frac{2}{3}}.$$

NOTE. **76.** Maclaurin's Formula has been demonstrated under the supposition, that in Equation (2) the coefficients are independent of x, and that the equation is

true for every possible value that can be attributed to x. If, then, the function u becomes infinite, when $x = 0$, the equation cannot be satisfied; neither can it be, if any one of the differential coefficients becomes infinite. Hence, any form of the function which produces either of these results, is excluded from the formula of Maclaurin. The functions,

$$u = \log x, \qquad u = \cot x, \qquad u = ax^{\frac{1}{2}},$$

are examples of such functions. In the first case, $u = -\infty$, when $x = 0$;* in the second, $u = \infty$, when $x = 0$; and in the third, B, and the succeeding differential coefficients, become infinite, when $x = 0$.

TAYLOR'S THEOREM.

77. TAYLOR'S THEOREM explains the method of developing into a series any function of the sum or difference of two independent variables.

78. Since the sum or difference of two independent variables may always be denoted by a single letter, any function of the form,

$$u' = f(x \pm y),$$

may be put under the form,

$$u' = f(z), \qquad \text{by making} \qquad z = x \pm y.$$

If we suppose z to be the abscissa, and u' the ordinate of a curve, and give to x an increment h, z will

* Bourdon, Art. **235**. University, Art. **186**. Legendre, Trig., Art. **22**

become $z + h$. If we pass to consecutive values, $dz = dx$, and

$$\frac{du'}{dz} = \frac{du'}{dx} = \tan \alpha. \quad \text{(Art. 13.)}$$

If we suppose x to remain constant, and y to receive the increment h, z will again become $z + h$, and when we pass to consecutive values,

$$\frac{du'}{dz} = \frac{du'}{dy} = \tan \alpha.$$

Hence, *in any function of the sum or difference of two independent variables, the partial differential coefficients are equal* (Art. **32**).

79. As an example, take,

$$u' = (x + y)^n.$$

If we suppose x to vary, the first partial differential coefficient is,

$$\frac{du'}{dx} = n(x + y)^{n-1}.$$

If we suppose y to vary, it is,

$$\frac{du'}{dy} = n(x + y)^{n-1};$$

and the same may be shown for the differential coefficients of the higher orders.

80. If any function of the form,

$$u' = f(x + y),$$

be developed into a series, it is plain that the series

must have terms containing the variables x and y, and that the constants, which enter into the given function, must also enter into the development.

Let us then assume,

$$f(u') = f(x + y) = A + By^a + Cy^b + Dy^c + \&c. \quad (1.)$$

in which the terms are arranged according to the ascending powers of y, and in which A, B, C, D, &c., are independent of y, but functions of x, and arbitrary functions of all the constants which enter the primitive function. It is now required to find such values for the exponents a, b, c, &c., and for the coefficients A, B, C, D, &c., as shall render the development true for all possible values that may be attributed to x and y.

In the first place, there can be no negative exponents. For, if any term were of the form,

$$By^{-a},$$

it might be written,

$$\frac{B}{y^a},$$

and making $y = 0$, this term would become infinite, and we should have,

$$f(x) = \infty,$$

which is absurd, since the function of x, which is independent of y, does not *necessarily* become infinite when $y = 0$.

The first term A, of the development, is the value which the primitive function u' assumes when we make $y = 0$.

If we designate this value by u, we shall have,

$$f(x) = u.$$

If we differentiate Equation (1), under the supposition that x varies, the partial differential coefficient is,

$$\frac{du'}{dx} = \frac{dA}{dx} + \frac{dB}{dx}y^a + \frac{dC}{dx}y^b + \frac{dD}{dx}y^c + \&c. \quad (2.)$$

and if we differentiate, regarding y as a variable, the partial differential coefficient is,

$$\frac{du'}{dy} = aBy^{a-1} + bCy^{b-1} + cDy^{c-1} + \&c. \quad (3.)$$

But these differential coefficients are equal to each other (Art. 78); hence, the second members of Equations (2) and (3) are equal. Since the coefficients are independent of y, and the equality exists whatever be the value of y, it follows that the corresponding terms in each series will contain like powers of y, and that the coefficients of y in these terms will be equal.* Hence,

$$a - 1 = 0, \quad b - 1 = a, \quad c - 1 = b, \quad \&c.,$$

and consequently,

$$a = 1, \quad b = 2, \quad c = 3, \quad \&c.$$

Comparing the coefficients, we find,

$$B = \frac{dA}{dx}, \quad C = \frac{1}{2}\frac{dB}{dx}, \quad D = \frac{1}{3}\frac{dC}{dx}.$$

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* Bourdon, Art. **195.** University, Art. **178.**

Since we have made,

$$f(x+y) = u', \qquad \text{and} \qquad f(x) = A = u,$$

we shall have,

$$A = u, \quad B = \frac{du}{dx}, \quad C = \frac{d^2u}{1.2dx^2}, \quad D = \frac{d^3u}{1.2.3dx^3},$$

and consequently,

$$u' = u + \frac{du}{dx}y + \frac{d^2u}{dx^2}\frac{y^2}{1.2} + \frac{d^3u}{dx^3}\frac{y^3}{1.2.3} + \text{\&c.},$$

which is the formula of Taylor.

In this formula, u is what u' becomes, when $y = 0$; $\frac{du}{dx}$, what $\frac{du'}{dx}$ becomes when $y = 0$; $\frac{d^2u}{dx^2}$, what $\frac{d^2u'}{dx^2}$ becomes when $y = 0$; and similarly for the other coefficients.

1. Let it be required to develop

$$u' = f(x+y)^n,$$

by this formula.

We find,

$$u = x^n, \quad \frac{du}{dx} = n.x^{n-1}, \quad \frac{d^2u}{dx^2} = n.(n-1)x^{n-2} + \text{\&c.};$$

hence,

$$u' = (x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1.2}x^{n-2}y^2$$

$$+ \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 + \text{\&c.}$$

SECTION V.

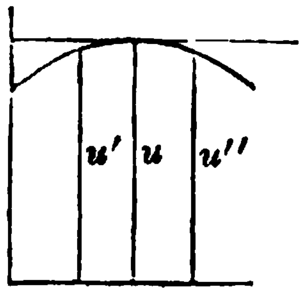
MAXIMA AND MINIMA.

81. A MAXIMUM value of a variable function is greater than the consecutive value which precedes, and the consecutive value which follows it.

A MINIMUM value of a variable function is less than the consecutive value which precedes, and the consecutive value which follows it.

If we denote any variable function by u, and the independent variable by x, every relation between u and x will be denoted by the co-ordinates of a curve whose equation is (Art. **10**),

$$u = f(x).$$



Let u' denote the consecutive ordinate which precedes u, and u'' the consecutive ordinate which follows it. Then, if u is a maximum,

$$u > u', \qquad \text{and} \qquad u > u'';$$

the curve therefore ascends *just before* the ordinate reaches a maximum value, and descends *immediately afterwards;* hence, at the point of maximum, it is concave towards the axis of abscissas (Art. **73**).

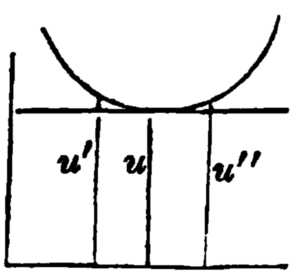
Since the curve *ascends* just before the ordinate reaches the maximum value, the first differential coefficient will be positive; and since it then *descends*, the first differential coefficient will be negative immediately after the

maximum value (Art. 72). Hence, at the *point of maximum* value of the ordinate, the first differential coefficient will change its sign, and therefore passes through 0.

Since the curve is concave towards the axis of abscissas, the second differential coefficient is negative (Art. 73); hence, the conditions of a maximum value of u are,

$$\frac{du}{dx} = 0, \qquad \text{and} \qquad \frac{d^2u}{dx^2}, \text{ negative.}$$

82. Denoting the consecutive ordinates, as before, by u', u, u'', if u is a minimum,



$$u < u', \qquad \text{and} \qquad u < u'';$$

the curve, therefore, descends *just before* the ordinate reaches a minimum, and ascends *immediately afterwards;* hence, at the point of minimum, it is convex towards the axis of abscissas.

Since the curve *descends* just before the ordinate reaches the minimum value, the first differential coefficient will be negative; and since it then *ascends*, the first differential coefficient will be positive immediately after the minimum value (Art. 72). Hence, at the point of *minimum value* of the ordinate, the first differential coefficient will change its sign, and therefore passes through 0.

Since the curve is convex towards the axis of abscissas, the second differential coefficient is positive (Art. 73); hence, the conditions of a minimum value of u, are,

$$\frac{du}{dx} = 0, \qquad \text{and} \qquad \frac{d^2u}{dx^2}, \text{ positive.}$$

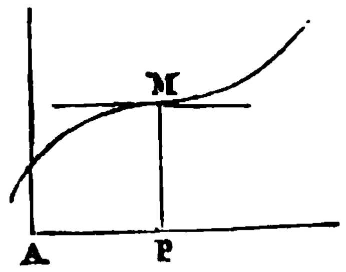
83. Hence, to find the maximum or minimum value of a function of a single variable:

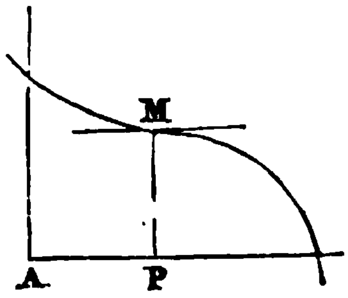
1. *Find the first differential coefficient of the function, place it equal to* 0, *and determine the roots of the equation.*

2. *Find the second differential coefficient, and substitute each real root, in succession, for the variable in the second member of the equation; each root which gives a negative result, will correspond to a maximum value of the function, and each which gives a positive result will correspond to a minimum value.*

Point of inflection.

84. A POINT OF INFLECTION is a point at which a curve changes its curvature with respect to the axis of abscissas.





When a curve is concave towards the axis of abscissas, its second differential coefficient is negative (Art. **72**); when it is convex, the second differential coefficient is positive (Art. **72**): therefore, at the point where the curve changes its curvature, the second differential coefficient changes its sign, and consequently passes through zero.

In the first figure, the second differential coefficient, at the point M, changes from negative to positive; in the second, from positive to negative. At the point M, in both figures, the first differential coefficient is equal to 0, and the tangent line separates the two branches

of the curve. When the second differential coefficient is 0, the ordinate at the point has neither a maximum nor a minimum.

There are three consecutive points of the curve which coincide with the tangent, at the point of inflection. This is shown by the equality of the co-ordinates of the point M (in the curve and tangent), and of the first and second differentials.

EXAMPLES.

1. To find the value of x which will render the function y a maximum or minimum in the equation of the circle,

$$y^2 + x^2 = R^2. \qquad \frac{dy}{dx} = -\frac{x}{y};$$

making, $\quad -\frac{x}{y} = 0, \quad$ gives, $\quad x = 0.$

The second differential coefficient is,

$$\frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^2}. \qquad \text{When, } x = 0, \quad y = R;$$

hence, $$\frac{d^2y}{dx^2} = -\frac{1}{R},$$

which being negative, y is a maximum.

2. Find the values of x which render the function y a maximum or minimum in the equation,

$$y = a - bx + x^2. \qquad \text{Differentiating,}$$

$$\frac{dy}{dx} = -b + 2x, \quad \text{and} \quad \frac{d^2y}{dx^2} = 2;$$

and making, $$-b + 2x = 0,$$

gives, $$x = \frac{b}{2}.$$

Since the second differential coefficient is positive, this value of x will render y a minimum. The minimum value of y is found by substituting the value of x, in the primitive equation. It is,

$$y = a - \frac{b^2}{4}.$$

3. Find the value of x which will render the function u a maximum or minimum in the equation,

$$u = a^4 + b^3x - c^2x^2.$$

$$\frac{du}{dx} = b^3 - 2c^2x, \qquad \text{hence,} \qquad x = \frac{b^3}{2c^2},$$

and $$\frac{d^2u}{dx^2} = -2c^2;$$

hence, the function is a maximum, and the maximum value is,

$$u = a^4 + \frac{b^6}{4c^2}.$$

4. Let us take the function,

$$u = 3a^2x^3 - b^4x + c^5.$$

We find, $$\frac{du}{dx} = 9a^2x^2 - b^4, \qquad \text{and} \qquad x = \pm\frac{b^2}{3a}.$$

The second differential coefficient is,

$$\frac{d^2u}{dx^2} = 18a^2x.$$

Substituting the plus root of x, we have,

$$\frac{d^2u}{dx^2} = +\,6ab^2,$$

which gives a minimum, and substituting the negative root, we have,

$$\frac{d^2u}{dx^2} = -\,6ab^2,$$

which gives a maximum.

The minimum value of the function is,

$$u = c^5 - \frac{2b^6}{9a};$$

and the maximum value,

$$u = c^5 + \frac{2b^6}{9a}.$$

5. Find the values of x, which make u a maximum or minimum in the equation,

$$u = x^5 - 5x^4 + 5x^3 - 1.$$

Ans. $\begin{cases} x = 1, & \text{a maximum.} \\ x = 3, & \text{a minimum.} \end{cases}$

6. Find the values of x, which make u a maximum or minimum in the equation,

$$u = x^3 - 9x^2 + 15x - 3.$$

Ans. $\begin{cases} x = -1, & \text{a maximum.} \\ x = +5, & \text{a minimum.} \end{cases}$

7. Find the values of x, which make u a maximum or minimum in the equation,

$$u = x^3 - 3x^2 + 3x + 7.$$

Ans. There is no such value of x, since the second differential coefficient reduces to 0, for $x = 1$; hence, only one condition of a maximum or minimum is fulfilled.*

85. NOTES. 1. In applying the preceding rules to practical examples, we first find an expression for the function which is to be made a maximum or minimum.

2. If in such expression, a constant quantity is found as a *factor*, it may be omitted in the operation; for the product will be a maximum or a minimum when the variable factor is a maximum or minimum.

3. Any value of the independent variable which renders a function a maximum or a minimum, will render any power or root of that function, a maximum or minimum; hence, we may square both members of an equation to free it of radicals, before differentiating.

8. To find the maximum rectangle which can be inscribed in a given triangle.

Let b denote the base of the triangle, h the altitude, y the base of the rectangle, and x its altitude. Then,

$$u = xy = \text{the area of the rectangle.}$$

But, $$b : h :: y : h - x;$$

hence, $$y = \frac{bh - bx}{h},$$

* We have limited the discussion to a single class of maxima and minima, viz.: that in which the first differential coefficient of the function is 0, and the second negative or positive.

and consequently,

$$u = \frac{bhx - bx^2}{h} = \frac{b}{h}(hx - x^2);$$

and omitting the constant factor $\frac{b}{h}$, we may write,

$$u' = hx - x^2;$$

for, the value of x, which makes u' a maximum, will make u a maximum (Art. 85); hence,

$$\frac{du'}{dx} = h - 2x, \qquad \text{or,} \qquad x = \frac{h}{2};$$

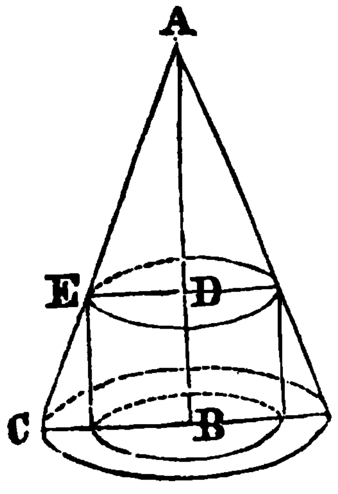
therefore, the altitude of the rectangle is equal to half the altitude of the triangle; and since,

$$\frac{d^2u'}{dx^2} = -2,$$

the area is a maximum (Art. 81).

9. What is the altitude of a cylinder inscribed in a given cone, when the volume of the cylinder is a maximum?

Suppose the cylinder to be inscribed, as in the figure, and let



$AB = a$, $BC = b$, $AD = x$, $ED = y$;

then, $BD = a - x =$ altitude of the cylinder, and

$\pi y^2(a - x)^* =$ volume $= v$ . . (1.)

From the similar triangles AED and ACB, we have,

* Legendre, Bk. VIII. Prop. 2.

$$x : y :: a : b; \qquad \text{whence,} \qquad y = \frac{bx}{a}.$$

Substituting this value in Equation (1), we have,

$$v = \frac{\pi b^2}{a^2} x^2(a - x).$$

Omitting the constant factor $\frac{\pi b^2}{a^2}$, we may write,

$$v' = x^2(a - x);$$

for, the conditions which will make v' a maximum, will also make v a maximum (Art. 85).

By differentiating, we have,

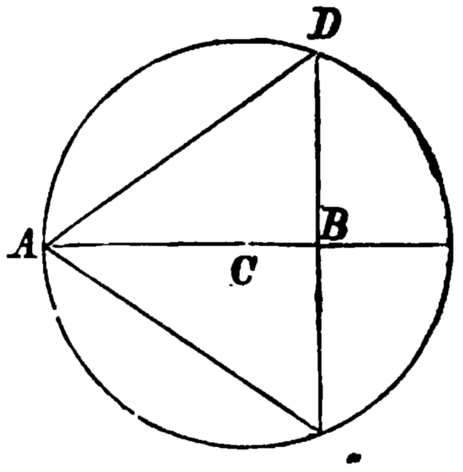
$$\frac{dv'}{dx} = 2ax - 3x^2.$$

Placing, $\qquad 2ax - 3x^2 = 0,$

we have, $\qquad x = 0, \qquad \text{and} \qquad x = \frac{2}{3}a.$

But, $\qquad \frac{d^2v'}{dx^2} = 2a - 6x = -2a.$

Hence, the cylinder is a maximum, when its altitude is one-third the altitude of the cone.



10. What is the altitude of a cone inscribed in a given sphere, when the volume is a maximum?

Denote the radius of the given sphere by r, and the centre by C. Let A be the vertex of the required cone, BD the radius of its base, which denote by y, and denote the altitude AB by x. Then,

$$y^2 = 2rx - x^2;^*$$

and if we denote the volume of the cone by v,

$$v = \tfrac{1}{3}\pi x(2rx - x^2) = \tfrac{1}{3}\pi(2rx^2 - x^3).\dagger$$

Omitting the constant factor $\frac{1}{3}\pi$, we have,

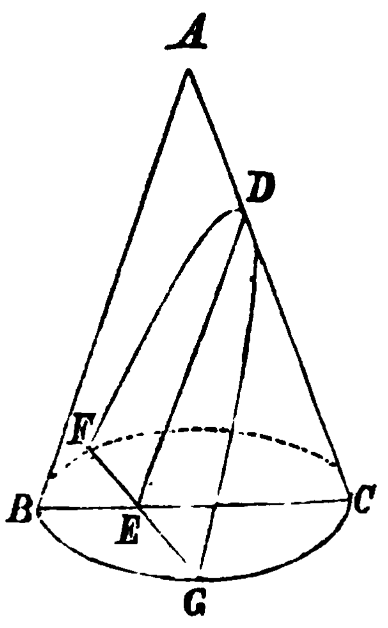
$$\frac{dv'}{dx} = 4rx - 3x^2; \qquad \text{hence,}$$

$$4rx - 3x^2 = 0, \qquad \text{and} \qquad x = \frac{4}{3}r;$$

that is, the altitude of the cone is four-thirds of the radius.

11. What is the altitude of a cone inscribed in a sphere when the convex surface is a maximum? *Ans.* $\frac{4}{3}r$.

12. What is the length of the axis of a maximum parabola which can be cut from a given right cone with a circular base?



Let BAC be a section of the cone by a plane passed through the axis; and FDG a parabola made by a plane parallel to the element BA.

Denote BC by b, AB by a, and CE by x; then, $BE = b - x$, and FE, the common ordinate of the circle and parabola, is equal to $\sqrt{bx - x^2}$.‡

* An. G., Bk. II. Art. 4—8. † Leg., Bk. VIII. Prop. 5.
‡ Legendre, Bk. IV. Prop. 25. Cor.

By similar triangles, we have,

$$b : a :: x : \frac{ax}{b} = DE.$$

Hence, the area of the parabola (Art. 59) is,

$$u = \frac{2}{3}\frac{ax}{b}\sqrt{bx - x^2}.$$

Omitting the constant factors, and remembering that the same value of x, which renders u a maximum, will render its square a maximum (Art. 79), and designating by u' the new function, we have,

$$u' = x^2(bx - x^2) = bx^3 - x^4, \qquad \text{and}$$

$$\frac{du'}{dx} = 3bx^2 - 4x^4; \quad \text{or,} \quad x = \frac{3}{4}b; \quad \text{and} \quad DE = \frac{3}{4}AB.$$

that is, the axis of the maximum parabola is three-fourths the slant height of the cone.

13. What is the altitude of the maximum rectangle which can be inscribed in a given parabola?

Ans. Two-thirds of the axis.

14. What are the sides of the maximum rectangle inscribed in a given circle?

Ans. A square whose side is $r\sqrt{2}$.

15. A cylindrical vessel, open at top, is to contain a given quantity of water. What is the relation between the radius of the base and the altitude, when the interior surface is a minimum?

Ans. Altitude = radius of base.

16. Required the maximum right-angled triangle which can be constructed on a given line, as a hypothenuse?

Ans. When it is isosceles.

17. Required the least triangle which can be formed by the two radii, produced, and a tangent line to the quadrant of a given circle? *Ans.* When it is isosceles.

18. What is the altitude of the maximum cylinder which can be inscribed in a given paraboloid?

Ans. Half the axis.

19. What is the altitude of a cylinder inscribed in a given sphere when its convex surface is a maximum?

Ans. $\frac{2r}{\sqrt{3}}$ (?)

20. What is the altitude of a cylinder inscribed in a given sphere, when its volume is a maximum?

Ans. $r\sqrt{2}$. (?)

21. Required the base of the maximum rectangle which can be inscribed in a given ellipse whose semi-axes are A and B.

Ans. $A\sqrt{2}$.

22. A rectangular sheep-fold, *to contain a given area*, is to be built against a wall. Required the ratio of the least side to the larger, so that the cost shall be a minimum.

Ans. 2.

23. To circumscribe a given circle whose radius is r, by an isosceles triangle whose area shall be a minimum.

Ans. Perpendicular to base $= 3r$.

SECTION VI.

DIFFERENTIALS OF TRANSCENDENTAL FUNCTIONS.

Differentials of Exponential and Logarithmic functions.

86. AN Exponential function is one in which the independent variable enters as an exponent; as,

$$u = a^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

If, in a function of this form, we give to x an increment h, we have,

$$u' = a^{x+h} = a^x a^h \quad . \quad . \quad . \quad . \quad (2.)$$

Subtracting Equation (1) from (2), member from member, we have,

$$u' - u = a^x a^h - a^x = a^x(a^h - 1);$$

whence,

$$\frac{u' - u}{a^x} = a^h - 1 \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Put, $a = 1 + b$, and develop by the binomial formula; we then have,

$$a^h = (1 + b)^h = 1 + hb + \frac{h}{1}\left(\frac{h-1}{2}\right)b^2 + \frac{h}{1}\left(\frac{h-1}{2}\right)\left(\frac{h-2}{3}\right)b^3$$

$$+ \frac{h}{1}\left(\frac{h-1}{2}\right)\left(\frac{h-2}{3}\right)\left(\frac{h-3}{4}\right)b^4 + \&c.$$

Substituting this value of a^h, in Equation (3), and dividing by h, we have,

$$\frac{u'-u}{a^x h} = b + \left(\frac{h-1}{2}\right)b^2 + \left(\frac{h-1}{2}\right)\left(\frac{h-2}{3}\right)b^3$$
$$+ \left(\frac{h-1}{2}\right)\left(\frac{h-2}{3}\right)\left(\frac{h-3}{4}\right)b^4 + \&c.$$

If we now pass to consecutive values, by making h numerically equal to 0, we have,

$$\frac{du}{a^x dx} = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \frac{b^5}{5} - \&c.;$$

and putting for b its value, $a - 1$, we have,

$$\frac{du}{a^x dx} = a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c. \quad (4.)$$

Denoting the second member of Equation (4) by k, we have,

$$\frac{du}{a^x dx} = k, \quad \text{or,} \quad du = da^x = a^x k dx \quad . \quad . \quad (5.)$$

that is, the differential of a function of the form a^x, *is equal to the function, into a constant quantity k, dependent on a, into the differential of the exponent.*

Relation between a and k.

87. The relation between a and k is very peculiar, and may be determined by Maclaurin's Formula,

$$u = a^x = \left(u\right) + \left(\frac{du}{dx}\right)x + \frac{1}{1.2}\left(\frac{d^2u}{dx^2}\right)x^2 + \frac{1}{1.2.3}\left(\frac{d^3u}{dx^3}\right)x^3$$
$$+ \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

First, if we make $x = 0$, the function $a^x = 1 = \left(u\right)$. The successive differential coefficients are found from Equation (5); viz.:

$$\frac{du}{dx} = a^x k, \quad \text{and} \quad \left(\frac{du}{dx}\right) = k;$$

$$d\left(\frac{du}{dx}\right) = \frac{d^2u}{dx} = da^x k = a^x k^2 dx; \text{ hence,}$$

$$\frac{d^2u}{dx^2} = a^x k^3, \quad \text{and} \quad \left(\frac{d^2u}{dx^2}\right) = k^2;$$

$$\frac{d^3u}{dx^3} = a^x k^3, \quad \text{and} \quad \left(\frac{d^3u}{dx^3}\right) = k^3,$$

&c., &c., &c.

Substituting these values in Equation (6), we have,

$$u = a^x = 1 + \frac{kx}{1} + \frac{k^2x^2}{1.2} + \frac{k^3x^3}{1.2.3} + \text{\&c.}$$

If we make $x = \frac{1}{k}$, we shall have,

$$a^{\frac{1}{k}} = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \text{\&c.};$$

designating the second member of the equation by e, and employing twelve terms of the series, we find,

$$e = 2.7182818\ldots.;$$

hence, $a^{\frac{1}{k}} = e$, therefore, $a = e^k$ . . (7.)

Equation (7) expresses the relation between a and k.

A system of logarithms, called the Naperian system, has been constructed, whose base is, $e = 2.7182818 \ldots.$ This, and the common system, whose base is 10, are the only systems in use. The logarithms, in the Naperian system, are denoted by l, and in the common system by log. We see from Equation (7), that k is the Naperian logarithm of the number a. If we take the common logarithms of both members of Equation (7), we shall have,

$$\log a = k \log e \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

The common logarithm of $e = \log 2.7182818 \ldots.$ $= .434284482 \ldots.,$ is called the modulus of the common system, and is denoted by M. Hence, *if we have the Naperian logarithm of a number, we can find the common logarithm of the same number by multiplying by the modulus.*

If, in Equation (8), we make $a = 10$, we have,

$$1 = k \log e; \qquad \text{or,} \qquad \frac{1}{k} = \log e = M;$$

that is, *the modulus of the common system is also equal to* 1, *divided by the Naperian logarithm of the common base.*

88. From Equation (5), we have,

$$\frac{du}{u} = \frac{da^x}{a^x} = kdx.$$

If we make $a = 10$, the base of the common system, $x = \log u$, and

$$dx = \frac{du}{u} \times \frac{1}{k} = \frac{du}{u} \times M;$$

that is, *the differential of a common logarithm of a quantity is equal to the differential of the quantity divided by the quantity into the modulus.*

89. If we make $a = e$, the base of the Naperian system, x becomes the Naperian logarithm of u, and k becomes 1: see Equation (7); hence, $M = 1$; and

$$dx = \frac{du}{a^x};$$

that is, *the differential of a Naperian logarithm of a quantity is equal to the differential of the quantity divided by the quantity; and in this system, the modulus is* 1.

90. Having found that k is the Naperian logarithm of a, we have from Equation (5),

$$du = a^x l\, a\, dx;$$

that is, *the differential of a function of the form a^x, is equal to the function, into the Naperian logarithm of the base a, into the differential of the exponent.*

EXAMPLES.

1. Find the differential of $u = a^x$.

$$du = a^x l\, a\, dx.$$

2. Find the differential of $u = l x$.

$$du = \frac{dx}{x} = x^{-1}dx.$$

NOTE. This case would seem to admit of integration by the rule of Art. **35**; but that rule applies to alge-

braic functions only, and this form is derived from a transcendental function.

3. Find the differential of $u = y^x$.

$$lu = xly; \text{ hence,}$$

$$\frac{du}{u} = x\frac{dy}{y} + ly\,dx; \text{ hence,}$$

by clearing of fractions, and reducing,

$$du = xy^{x-1}dy + y^x ly\,dx;$$

that is, equal to the sum of the partial differentials (Art. 32).

4. Find by logarithms the differential of $u = xy$.

$$lu = lx + ly;^* \text{ hence,}$$

$$\frac{du}{u} = \frac{dx}{x} + \frac{dy}{y}; \text{ and by reducing,}$$

$$du = ydx + xdy \quad \text{(Art. 27).}$$

5. Find by logarithms the differential of $u = \frac{x}{y}$.

$$lu = lx - ly;\dagger \text{ hence, by differentiating,}$$

$$\frac{du}{u} = \frac{dx}{x} - \frac{dy}{y}; \text{ and by reducing,}$$

$$du = \frac{ydx - xdy}{y^2} \quad \text{(Art. 29).}$$

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* Bourdon, Art. 230. University, Art. 185.

† Bourdon, Art. 231. University, Art. 185.

6. Find the differential of $u = l\left(\frac{x+a}{a-x}\right)$.

$$du = \frac{2a\,dx}{a^2 - x^2}.$$

7. Find the differential of $u = l\left(\frac{x}{\sqrt{a^2+x^2}}\right)$.

$$du = \frac{a^2dx}{x(a^2+x^2)}.$$

8. Find the differential of $u = (a^x + 1)^2$.

$$du = 2a^x(a^x + 1)\,l\,a\,dx.$$

9. Find the differential of $u = \frac{a^x - 1}{a^x + 1}$.

$$du = \frac{2a^x\,l\,a\,dx}{(a^x+1)^2}.$$

10. Find the differential of $u = \frac{a^x}{x^x} = \left(\frac{a}{x}\right)^x$.

$$du = \left(\frac{a}{x}\right)^x\left(l\frac{a}{x} - 1\right)dx.$$

Differential forms which have known integrals.

91. If we have a differential in a fractional form, in which the numerator is the differential of the denominator, we know that the integral is the Naperian logarithm of the denominator (Art. **89**). It frequently happens, however, that we have to deal with fractional differentials which are not of this form, but which, by certain algebraic artifices, may be reduced to it. We shall give a few examples of such reductions.

Form 1. $$\int \frac{dx}{\sqrt{x^2 \pm a^2}}.$$

Put $x^2 \pm a^2 = v^2$; then, $xdx = vdv$.

Add vdx to both members; then,

$$xdx + vdx = vdx + vdv; \quad \text{hence,}$$

$$(x + v)dx = v(dx + dv); \quad \text{whence,}$$

$$\frac{dx + dv}{x + v} = \frac{dx}{v} = \frac{dx}{\sqrt{x^2 \pm a^2}}; \quad \text{hence,}$$

$$\int \frac{dx + dv}{x + v} = \int \frac{dx}{\sqrt{x^2 \pm a^2}}.$$

But in the first member, the numerator is the differential of the denominator; hence,

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = l(x + v) = l\left(x + \sqrt{x^2 \pm a^2}\right).$$

Form 2. $$\int \frac{dx}{\sqrt{x^2 \pm 2ax}}.$$

Put $\sqrt{x^2 \pm 2ax} = v$; then, $x^2 \pm 2ax = v^2$.

Adding a^2 to both members, and extracting the square root,

$$x \pm a = \sqrt{v^2 + a^2}; \quad \text{hence,} \quad dx = \frac{vdv}{\sqrt{v^2 + a^2}},$$

and $$\frac{dx}{\sqrt{x^2 \pm 2ax}} = \frac{dv}{\sqrt{v^2 + a^2}}.$$

But from the first form,

$$\int \frac{dv}{\sqrt{v^2 + a^2}} = l\left(v + \sqrt{v^2 + a^2}\right).$$

Substituting for v its value, and for $\sqrt{v^2 + a^2}$, its value,

$$\int \frac{dx}{\sqrt{x^2 \pm 2ax}} = l\left(x \pm a + \sqrt{x^2 \pm 2ax}\right).$$

Form 3. $\dfrac{2adx}{a^2 - x^2}$; or, $\dfrac{2adx}{x^2 - a^2}$.

Since, $$\frac{2adx}{a^2 - x^2} = \frac{2adx}{(a + x)(a - x)} = \frac{dx}{a + x} + \frac{dx}{a - x}.^*$$

$$\int\left(\frac{dx}{a + x} + \frac{dx}{a - x}\right) = \int \frac{dx}{a + x} + \int \frac{dx}{a - x}$$

$$\int \frac{2adx}{a^2 - x^2} = l(a + x) - l(a - x) = l\left(\frac{a + x}{a - x}\right).$$

Also, $$\int \frac{2adx}{x^2 - a^2} = l\left(\frac{x - a}{x + a}\right).$$

(See Example 6, page 126.)

Form 4. $$\frac{2adx}{x\sqrt{a^2 \pm x^2}}.$$

Put $\sqrt{a^2 + x^2} = v$; whence, $a^2 + x^2 = v^2$; hence,

$$x^2 = v^2 - a^2, \text{ and } xdx = vdv, \quad \text{or,} \quad dx = \frac{vdv}{x}.$$

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* University, Art. **180.** (See Art. **158.**)

Multiply both members by $\frac{2a}{x\sqrt{a^2+x^2}}$; we have,

$$\int\frac{2adx}{x\sqrt{a^2+x^2}} = \int\frac{2adv}{v^2-a^2} = l\left(\frac{v-a}{v+a}\right); \text{ hence,}$$

$$\int\frac{2adx}{x\sqrt{a^2+x^2}} = l\left(\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}\right).$$

In like manner we should find,

$$\int\frac{2adx}{x\sqrt{a^2-x^2}} = l\left(\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}\right).$$

Form 5. $$\int\frac{x^{-2}dx}{\sqrt{a^2+x^{-2}}}.$$

Put $\frac{1}{x} = v$; then, $x^{-2}dx = -dv$; and

$$\frac{x^{-2}dx}{\sqrt{a^2+x^{-2}}} = \frac{-dv}{\sqrt{a^2+v^2}}; \text{ first form.}$$

$$\int\frac{x^{-2}dx}{\sqrt{a^2+x^{-2}}} = -l\left(v+\sqrt{a^2+v^2}\right)$$

$$= -l\left(\frac{1}{x}+\sqrt{a^2+\frac{1}{x^2}}\right)$$

$$= -l\left(\frac{1+\sqrt{1+a^2x^2}}{x}\right).$$

TABLE OF FORMS.

1. $\int a^x \, l a \, dx = a^x$ . . . . (Ex. 1.)

2. $\int \frac{dx}{x} = dx\, x^{-1} = l x$ . . . . . (Ex. 2.)

3. $\int (xy^{x-1} dy + y^x \, l y + dx) = y^x$. (Ex. 3.)

4. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = l\left(x + \sqrt{x^2 \pm a^2}\right)$. (Form 1.)

5. $\int \frac{dx}{\sqrt{x^2 \pm 2ax}} = l\left(x \pm a + \sqrt{x^2 \pm 2ax}\right)$. (2.)

6 $\int \frac{2a dx}{a^2 - x^2} = l\left(\frac{a + x}{a - x}\right)$. (Form 3.)

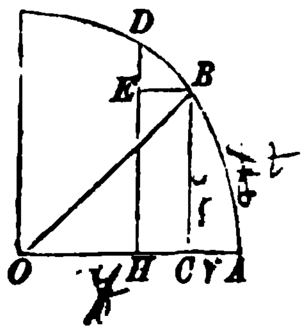
7. $\int \frac{2a dx}{x^2 - a^2} = l\left(\frac{x - a}{x + a}\right)$. (Form 3.)

8. $\int \frac{2a dx}{x\sqrt{a^2 + x^2}} = l\left(\frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a}\right)$. (Form 4.)

9. $\int \frac{2a dx}{x\sqrt{a^2 - x^2}} = l\left(\frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}}\right)$. (Form 4.)

10. $\int \frac{x^{-2} dx}{\sqrt{a^2 + x^{-2}}} = -\, l\left(\frac{1 + \sqrt{1 + a^2x^2}}{x}\right)$. (5.)

CIRCULAR FUNCTIONS.



92. LET O be the centre of a circle, A the origin of arc, and BC, DF any two consecutive ordinates. Draw BE parallel to OA; draw the radius OB, and denote it by 1. Denote the arc AB by z, and suppose z to be the independent variable. Then, BD will be the differential of the arc AB; ED, the differential of the sine, and EB the differential of the cosine, which will be negative, since it is a decreasing function of the arc (Art. **19**).

93. Since the triangles OBC and DEB, have their sides respectively perpendicular to each other, they will be similar;* hence,

$$OB : OC :: BD : DE; \text{ or,}$$

$$1 : \cos z :: dz : d\sin z; \text{ whence,}$$

$$d \sin z = \cos z\, dz \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

94. Again, $1 : \sin z :: dz : -d\cos z$; whence,

$$d \cos z = -\sin z\, dz \quad . \quad . \quad . \quad . \quad (2.)$$

95. Since,

$$\cos z = 1 - \text{ver-sin}\, z, \qquad d\cos z = -d \;\text{ver-sin}\, z;$$

hence, $$d\;\text{ver-sin}\, z = \sin z\, dz \quad . \quad . \quad . \quad . \quad (3.)$$

* Legendre, Bk. IV. Prop. 21.

96. Again, $\tan z = \frac{\sin z}{\cos z}$; hence,

$$d \tan z = \frac{\cos z\, d \sin z - \sin z\, d \cos z}{\cos^2 z} \text{ (Art. 29).}$$

Substituting for $d \sin z$ and $d \cos z$, their values from Equations (1) and (2), we have,

$$d \tan z = \frac{dz}{\cos^2 z} \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

By similar processes, we can find the differentials of the co-versed-sine, cotangent, secant, and cosecant, in terms of the other functions and the differential of z.

97. Denote the sine of the arc AB by y, its cosine by x, its versed sine by v, and its tangent by t. If we regard each of these as the independent variable, and the arc z as the common function, and find the values of z from Equations (1), (2), (3), and (4), we shall have,

When radius $= 1$,	When radius $= r$,
$dz = \frac{dy}{\sqrt{1 - y^2}} \quad . \quad . \quad . \quad (5.)$	$dz = \frac{r\,dy}{\sqrt{r^2 - y^2}} \quad . \quad . \quad . \quad (9.)$
$dz = -\frac{dx}{\sqrt{1 - x^2}} \quad . \quad . \quad (6.)$	$dz = -\frac{r\,dx}{\sqrt{r^2 - x^2}} \quad . \quad (10.)$
$dz = \frac{dv}{\sqrt{2v - v^2}} \quad . \quad . \quad . \quad (7.)$	$dz = \frac{r\,dv}{\sqrt{2rv - v^2}} \quad . \quad . \quad (11.)$
$dz = \frac{dt}{1 + t^2} \quad . \quad . \quad . \quad . \quad . \quad (8.)$	$dz = \frac{r^2 dt}{r^2 + t^2} \quad . \quad . \quad . \quad . \quad . \quad (12.)$

The differential of the arc, in terms of either of the other functions is easily found.

98. The following notation is employed to designate an arc by means of any one of its functions.

$\sin^{-1}u$, denotes the arc of which u is the sine,

$\cos^{-1}u$, denotes the arc of which u is the cosine,

$\tan^{-1}u$, denotes the arc of which u is the tangent,

&c., &c., &c.

If we denote the sine of an arc by $\frac{u}{a}$, instead of y, as in Equation (5), we shall have,

$$y = \frac{u}{a}, \qquad dy = \frac{du}{a}, \qquad \text{and} \qquad z = \sin^{-1}\frac{u}{a}.$$

Substituting these values in Equation (5), we have,

$$dz = \frac{du}{\sqrt{a^2 - u^2}} \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

Denoting the cosine of the arc by $\frac{u}{a}$, and making like substitutions in Equation (6), we have,

$$dz = \frac{-\,du}{\sqrt{a^2 - u^2}} \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

Denoting the ver-sine of the arc by $\frac{u}{a}$, and making like substitutions in Equation (7), we have,

$$dz = \frac{du}{\sqrt{2u - u^2}} \quad . \quad . \quad . \quad . \quad (15.)$$

Denoting the tangent of an arc by $\frac{u}{a}$, we have from Equation (8),

$$dz = \frac{adu}{1+u^2} \quad . \quad . \quad . \quad . \quad . \quad (16.)$$

EXAMPLES.

1. Differentiate the function,

$$z = \cos^{-1}\left(u\sqrt{1-u^2}\right).$$

$$dz = \frac{(-1+2u^2)du}{\sqrt{(1-u^2+u^4)(1-u^2)}}.$$

2. Differentiate the function,

$$z = \sin^{-1}\left(2u\sqrt{1-u^2}\right). \qquad dz = \frac{2du}{\sqrt{1-u^2}}.$$

3. Differentiate the function,

$$z = \tan^{-1}\frac{x}{y}, \qquad dz = \frac{ydx - xdy}{y^2+x^2}.$$

4. Differentiate the function,

$$z = \cos x^{\sin x}.$$

Make, $\cos x = u$, and $\sin x = y$;

then, $z = u^y$, and, (Art. 90),

$$dz = u^y\, l\, u\, dy + yu^{y-1}\, du;$$

also, $du = -\sin x\, dx$, and $dy = \cos x\, dx$;

hence,
$$dz = u^v\left(lu\,dy + \frac{y}{u}\,du\right)$$
$$= \cos x^{\sin x}\left(l\cos x\cos x - \frac{\sin^2 x}{\cos x}\right)dx.$$

Differential forms which have known Integrals.

99. The first four equations in Art. **92** furnish us four forms, by taking the integrals of both members. Equations (5), (6), (7), and (8), are of the same form as Equations (9), (10), (11), and (12), except that the radius is 1 in the first set, and r in the second; hence, the arc z, in each equation of the second set, is r times as great as in the corresponding equation of the first set.*

Forms (13), (14), (15), and (16), are modified forms of (5), (6), (7), and (8). They differ from them only in the symbol by which the function of the arc is denoted.

TABLE OF FORMS.

1. $$\int \cos z\,dz = \sin z + C.$$

2. $$\int -\sin z\,dz = \cos z + C.$$

3. $$\int \sin z\,dz = \text{ver-sin}\,z + C.$$

4. $$\int \frac{dz}{\cos^2 z} = \tan z + C.$$

5. $$\int \frac{dy}{\sqrt{1-y^2}} = \sin^{-1} y + C.$$

* Leg., Trig. Art. **41**.

6. $$\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1}x + C.$$

7. $$\int \frac{dv}{\sqrt{2v-v^2}} = \text{ver-sin}^{-1}v + C.$$

8. $$\int \frac{dt}{1+t^2} = \tan^{-1}t + C.$$

$$\left.\begin{array}{lll}
9. & \displaystyle\int \frac{rdy}{\sqrt{r^2-y^2}} & = \sin^{-1}y + C. \\[2ex]
10. & \displaystyle\int \frac{-rdx}{\sqrt{r^2-x^2}} & = \cos^{-1}x + C. \\[2ex]
11. & \displaystyle\int \frac{rdv}{\sqrt{2v+v^2}} & = \text{ver-sin}^{-1}v + C. \\[2ex]
12. & \displaystyle\int \frac{r^2dt}{r^2+t^2} & = \tan^{-1}t + C.
\end{array}\right\} \text{To radius } r.$$

13. $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}.$$

14. $$\int \frac{-du}{\sqrt{a^2-u^2}} = \cos^{-1}\frac{u}{a}.$$

15. $$\int \frac{du}{\sqrt{2au-u^2}} = \text{ver-sin}^{-1}\frac{u}{a}.$$

16. $$\int \frac{adu}{a^2+u^2} = \tan^{-1}\frac{u}{a}.$$

Applications.

100. We may readily find the relation between the diameter and the circumference of a circle from either of the first four equations of Art. **97**.

1. To find this ratio from Equation (5), which is,

$$dz = \frac{dy}{\sqrt{1-y^2}}; \quad \text{or,} \quad \frac{dz}{dy} = \frac{1}{\sqrt{1-y^2}} = (1-y^2)^{-\frac{1}{2}}.$$

Developing by the Binomial Formula, we have,

$$\frac{dz}{dy} = 1 + \frac{1}{2}y^2 + \frac{1.3}{2.4}y^4 + \frac{1.3.5}{2.4.6}y^6 + \&c.; \quad \text{whence,}$$

$$dz = dy + \frac{1}{2}y^2dy + \frac{1.3}{2.4}y^4dy + \frac{1.3.5}{2.4.6}y^6dy + \&c.$$

$$\int dz = z = y + \frac{1}{2.3}y^3 + \frac{1.3}{2.4.5}y^5 + \frac{1.3.5}{2.4.6.7}y^7 + \&c.$$

If we make $z = 30°$, of which the sine y is $\frac{1}{2}$,* we have,

$$30° = \frac{1}{2} + \frac{1}{2.3.2^3} + \frac{1.3}{2.4.5.2^5} + \frac{1.3.5}{2.4.6.7.2^7} + \&c.$$

By multiplying both members of the equation by 6, and taking twelve terms of the series, we have,

$$180° = \pi = 3.1415924,$$

which is true to the last place, which should be 6.

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* Legendre, Trig. Art. **29**.

2. Find the ratio from Equation (8), which is,

$$dz = \frac{dt}{1+t^2}; \quad \text{or,} \quad \frac{dz}{dt} = \frac{1}{1+t^2} = (1+t^2)^{-1}.$$

Developing by the Binomial Formula, we have,

$$\frac{dz}{dt} = 1 - t^2 + t^4 - t^6 + t^8 - \text{\&c.}; \quad \text{whence,}$$

$$dz = dt - t^2dt + t^4dt - t^6dt + t^8dt - \text{\&c.}$$

$$\int dz = z = \tan^{-1}t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \text{\&c.}$$

This series is not sufficiently converging. To find the value of the arc in a more converging series, we employ the following property of two arcs, viz.:

Four times the arc whose tangent is $\frac{1}{5}$, *exceeds the arc of* 45° *by the arc whose tangent is* $\frac{1}{239}$.*

* Let a denote the arc whose tangent is $\frac{1}{5}$. Then, Leg., Trig. Art. **36**.,

$$\tan 2a = \frac{2\tan a}{1-\tan^2 a} = \frac{5}{12},$$

$$\tan 4a = \frac{2\tan 2a}{1-\tan^2 2a} = \frac{120}{119}.$$

The last number being greater than 1, shows that the arc $4a$ exceeds 45°. Making,

$$4a = A, \qquad 45° = B,$$

But, $\tan^{-1}\left(\frac{1}{5}\right) = \frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} + \text{\&c.},$

$$\tan^{-1}\left(\frac{1}{239}\right) = \frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} + \text{\&c.}$$

hence

$$\text{arc } 45^\circ = \left\{ \begin{array}{l} 4\left(\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.7^7} +\right) \\ \\ -\left(\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} +\right) \end{array} \right\}$$

Multiplying both members by 4, we find,

$$180^\circ = \pi = 3.141592653.$$

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the difference, $4a - 45^\circ = A - B = b$, will have for its tangent,

$$\tan b = \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{1}{239};$$

hence, *four times the arc whose tangent is* $\frac{1}{5}$, *exceeds the arc of* 45° *by an arc whose tangent is* $\frac{1}{239}$.

SECTION VII.

TRANSCENDENTAL CURVES — CURVATURE — RADIUS OF CURVATURE — INVOLUTES AND EVOLUTES.

Classification of Curves.

101. Curves may be divided into two general classes: 1st. Those whose equations are purely algebraic; and 2dly. Those whose equations involve transcendental quantities.

Those of the first class, are called Algebraic curves, and those of the second, *Transcendental curves.*

The properties of the Algebraic curves have been already examined; it therefore only remains to explain the properties of the Transcendental curves.

Logarithmic Curve.

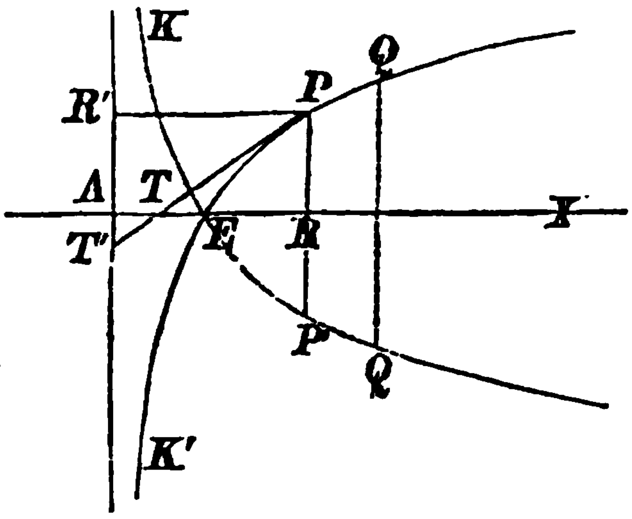
102. A logarithmic curve, is a curve in which one of the co-ordinates, of any point, is the logarithm of the other. The co-ordinate axis to which the lines denoting the logarithms are parallel, is called the *axis of logarithms*, and the other, the *axis of numbers.*

If we suppose Y to be the axis of logarithms, then X will be the axis of numbers, and the equation of the curve will be,

$$y = \log x.$$

General Properties.

103. Let A be the origin of a system of rectangular co-ordinates, X the axis of numbers, and Y the axis of logarithms.



If we designate the base of a system of logarithms by a, we shall have,*

$$a^y = x,$$

in which y is the logarithm of x.

If we change the value of the base a, to a', we shall have,

$$a'^y = x,$$

in which y is the logarithm of x, to the base a'. It is plain, that the same value of x, in the two equations, will give different values of y, and hence: *Each system of logarithms will give a different logarithmic curve.*

If we make $y = 0$, we shall have,† $x = 1$; and since this relation is independent of the base of the system of logarithms, it follows, that: *Every logarithmic curve will intersect the axis of numbers at a distance from the origin equal to* 1.

This abscissa is denoted by the line AE.

We may find points of the curve from the general equation,

$$a^y = x,$$

* Bourdon, Art. **227.** University, Art. **183.**
† Bourdon, Art. **235.** University, Art. **186.**

even without the aid of a table of logarithms. For, if we make,

$$y = 0, \qquad y = \frac{1}{2}, \qquad y = \frac{3}{2}, \qquad y = \frac{1}{4}, \quad \&c.,$$

we shall find, for the corresponding values of x,

$$x = 1, \qquad x = \sqrt{a}, \qquad x = a\sqrt{a}, \qquad x = \sqrt[4]{a}, \quad \&c.$$

If we make $a = 10$, the curve will correspond to the common system of logarithms; and if we suppose $a = 2.7182818\ldots$, to the Naperian system. Both curves pass through the point E.

Base > 1.

104. If we suppose the base of the system of logarithms to be greater than 1, the logarithms of all numbers less than 1 will be negative;* therefore, the values of y, corresponding to all abscissas between the limits of $x = 0$, and $x = AE = 1$, will be negative; hence, these ordinates are laid off below the axis of X. When $x = 0$, $y = -\infty$. When the base is greater than 1, the corresponding curve is $QPEK'$. The curve cannot extend to the left of the axis of Y, since negative numbers have no real logarithms.†

Base < 1.

105. If the base of the system is less than 1, the logarithms of all numbers greater than 1 are negative; and of all numbers less than 1, positive. Under this supposition, the curve assumes the position $Q'P'EK$. The parts

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* Bourdon, Art. **235.** University, Art. **186.**
† Bourdon, Art. **235.** University, Art. **186**

of the curves EPQ, $EP'Q'$, are concave towards the axis of abscissas; the parts EK, EK', are convex; and both curves, throughout their whole extent, are convex towards the axis of Y.

Asymptote.

106. Let us resume the equation of the curve,

$$y = \log x.$$

If we denote the modulus of a system of logarithms by M, and differentiate, we have (Art. **88**),

$$dy = \frac{dx}{x}M; \qquad \text{or,} \qquad \frac{dy}{dx} = \frac{M}{x}.$$

But, $\dfrac{dy}{dx}$ denotes the tangent of the angle which the tangent line makes with the axis of abscissas; hence, the tangent will be parallel to the axis of abscissas when $x = \infty$, and perpendicular to it, when $x = 0$.

But, when $x = 0$, $y = -\infty$; hence, the axis of ordinates is an asymptote to the curve. The tangent which is parallel to the axis of X, is not an asymptote; for, when $x = \infty$, we also have, $y = \infty$ (Art. **50**).

Sub-tangent.

107. The most remarkable property of this curve, is the value of its sub-tangent $T'R'$, estimated on the axis of logarithms. We have found, for the sub-tangent, on the axis of X (Art. **45**),

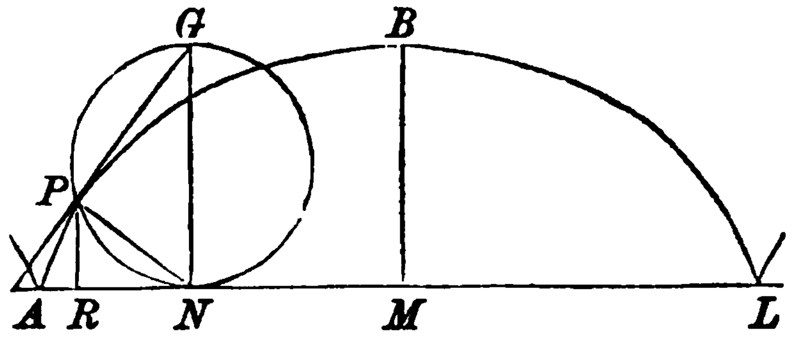
$$TR = \frac{dx}{dy}y;$$

and by simply changing the axis, we have,

$$T'R' = \frac{dy}{dx}x = M \text{ (Art. 106)}; \qquad \text{hence,}$$

The sub-tangent, taken on the axis of logarithms, is equal to the modulus of the system from which the curve is constructed. In the Naperian system, $M = 1$; hence, the sub-tangent is equal to 1, equal to AE. In the common system, it is denoted by the number, .434284482...

The Cycloid.



108. If a circle NPG be rolled along a straight line, AL, any point of the circumference, as P, will describe a curve, which is called a *cycloid.* The circle NPG is called the *generating circle*, and P, the *generating point.*

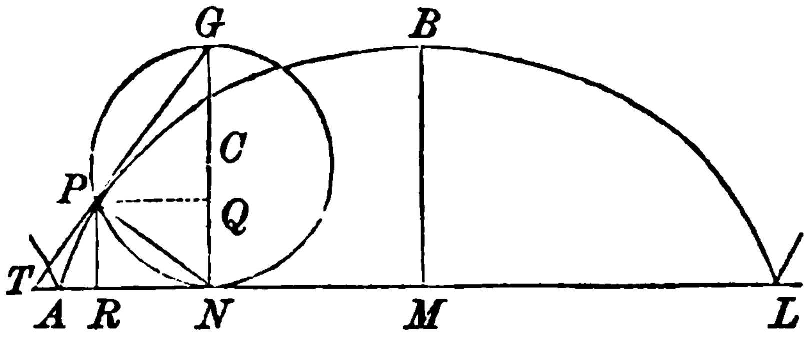
Since each succeeding revolution of the generating circle will describe an equal curve, it will only be necessary to examine the properties of the curve $APBL$, described in one revolution. We shall, therefore, refer only to this part, when speaking of the cycloid.

If we suppose the point P to be on the line AL, at A, it will be found at some point, as L, after all the points of the circumference shall have been brought in contact with the line AL. The line AL will be equal to the circumference of the generating circle, and is called the

base of the cycloid. The line BM, drawn perpendicular to the base, at the middle point, is called the *axis of the cycloid*, and is equal to the diameter of the generating circle.

Transcendental Equation of the Cycloid.

109. Let CN be the radius of the generating circle. Assume any point, as A, for the origin of co-ordinates. Let us suppose that when the generating point has described any arc of the cycloid, as AP, that the point in which the circle touches the base has reached the point N.



Through N, draw the diameter NG, of the generating circle: it will be perpendicular to the base AL. Through P, draw PR perpendicular to the base, and PQ parallel to it. Then, $PR = NQ$ will be the versed sine, and PQ the sine of the arc NP to the radius CN. Put,

$$CN = r, \quad AR = x, \quad PR = NQ = y;$$

we shall then have,

$$PQ = \sqrt{2ry - y^2}, \quad x = AN - RN = \text{arc } NP - PQ;$$

hence, the transcendental equation is,

$$x = \text{ver-sin}^{-1} y - \sqrt{2ry - y^2}.$$

Differential Equation.

110. The properties of the cycloid are most easily deduced from its differential equation. This is found by differentiating both members of the transcendental equation. We have (Art. **97**),

$$d(\text{ver-sin}^{-1}y) = \frac{rdy}{\sqrt{2ry - y^2}}; \text{ and}$$

$$d(-\sqrt{2ry - y^2}) = -\frac{rdy - ydy}{\sqrt{2ry - y^2}}; \text{ hence,}$$

$$dx = \frac{rdy}{\sqrt{2ry - y^2}} - \frac{rdy - ydy}{\sqrt{2ry - y^2}}; \quad \text{or,} \quad dx = \frac{ydy}{\sqrt{2ry - y^2}};$$

which is the differential equation of the cycloid.

Sub-Tangent, Tangent, Sub-Normal, Normal.

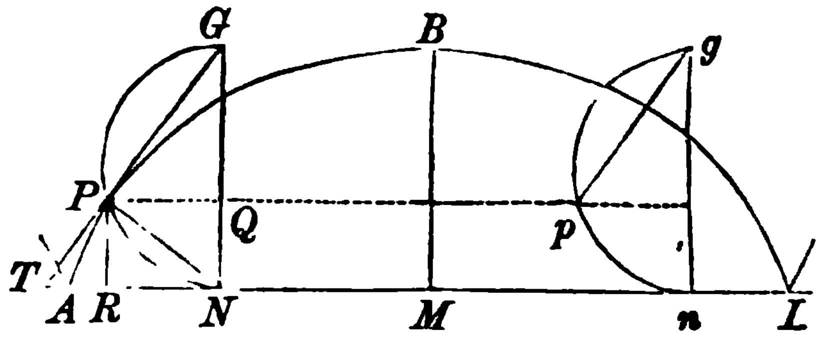
111. If we substitute in the general equations of Arts. **45**, **46**, **47**, and **48**, the values of dx and dy, found in the differential equation of the cycloid, we shall obtain the values of the sub-tangent, tangent, normal, and sub-normal. They are,

$$TR = \frac{y^2}{\sqrt{2ry - y^2}} = \text{sub-tangent};$$

$$TP = \frac{y\sqrt{2ry}}{\sqrt{2ry - y^2}} = \text{tangent};$$

$$PN = \sqrt{2ry} = \text{normal};$$

$$RN = \sqrt{2ry - y^2} = \text{sub-normal}.$$



These values are easily constructed, from their connection with the parts of the generating circle.

The sub-normal RN, for example, is equal to PQ of the generating circle, since each is equal to $\sqrt{2ry - y^2}$; hence, the normal PN, and the diameter GN, intersect the base of the cycloid at the same point. Now, since the tangent to the cycloid at the point P must be perpendicular to the normal, it will coincide with the chord PG of the generating circle.

If, therefore, it be required to draw a normal, or a tangent, to the cycloid, at any point, as P, draw any line, as ng, perpendicular to the base AL, and make it equal to the diameter of the generating circle. On ng, describe a semi-circumference, and through P draw a parallel to the base of the cycloid. Through p, where the parallel cuts the semi-circumference, draw the supplementary chords pn, pg, and then draw through P the parallels PN, PG; and PN will be a normal, and PG a tangent to the cycloid at the point P.

Position of Tangent.

112. The differential equation of the curve,

$$dx = \frac{ydy}{\sqrt{2ry - y^2}},$$

may be put under the form,

$$\frac{dy}{dx} = \frac{\sqrt{2ry - y^2}}{y} = \sqrt{\frac{2r}{y} - 1}.$$

If we make $y = 0$, we shall have,

$$\frac{dy}{dx} = \infty;$$

and if we make $y = 2r$, we shall have,

$$\frac{dy}{dx} = 0;$$

hence, the tangent lines drawn to the cycloid at the points where the curve meets the base, are perpendicular to the base; and the tangent drawn through the extremity of the greatest ordinate, is parallel to the base.

Curve Concave.

113. If we differentiate the equation,

$$dx = \frac{ydy}{\sqrt{2ry - y^2}},$$

regarding dx as constant, we obtain,

$$0 = (yd^2y + dy^2)\sqrt{2ry - y^2} - \frac{ydy(rdy - ydy)}{\sqrt{2ry - y^2}};$$

or, by reducing and dividing by y,

$$0 = (2ry - y^2)d^2y + rdy^2,$$

whence we obtain,

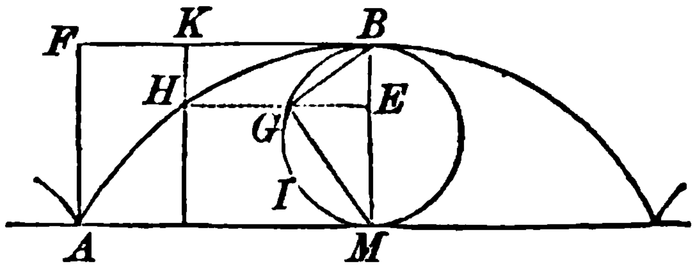
$$d^2y = - \frac{rdy^2}{2ry - y^2};$$

and hence, the curve is concave towards the axis of abscissas (Art. 73).

Area of the Cycloid.

114. The area of the cycloid may be found in a very simple manner, by constructing the rectangle $AFBM$, and considering the portion AFB.

If we regard F as an origin of co-ordinates, FB as a line of abscissas, and take any ordinate, as,



$$KH = z = 2r - y,$$

we shall have, $$d(AHKF) = zdx.$$

But, $$zdx = \frac{(2r - y)ydy}{\sqrt{2ry - y^2}} = dy\sqrt{2ry - y^2};$$

whence, $$AHKF = \int dy\sqrt{2ry - y^2} + C.$$

But this integral expresses the area of the segment of a circle, whose radius is r, and versed-sine y (Art. 99), that is, of the segment $MIGE$. If now, we estimate the area of the segment from M, where $y = 0$, and the area $AFKH$ from AF, in which case the area $AFKH = 0$, for $y = 0$, we shall have,

$$AFKH = MIGE;$$

and taking the integral between the limits $y = 0$ and $y = 2r$, we have,

$$AFB = \text{semi-circle } MIGB,$$

and consequently,

$$\text{area } AHBM = AFBM - MIGB.$$

But the base of the rectangle $AFBM$ is equal to the semi-circumference of the generating circle, and the altitude is equal to the diameter; hence, its area is equal to four times the area of the semi-circle $MIGB$; therefore,

$$\text{area } AHBM = 3MIGB; \text{ hence,}$$

The area AHBL is equal to three times the area of the generating circle.

Surface described by the Cycloid.

115. To find the surface described by the arc of a cycloid when revolved about its base.

The differential equation of the cycloid is,

$$dx = \frac{ydy}{\sqrt{2ry - y^2}}.$$

Substituting this value of dx in the differential equation of the surface (Art. **62**), it becomes,

$$ds = \frac{2\pi\sqrt{2r}\,y^{\frac{3}{2}}dy}{\sqrt{2ry - y^2}}.$$

Applying Formula (E), (Art. **170**), we have,

$$s = 2\pi\sqrt{2r}\left[-\frac{2}{3}y^{\frac{1}{2}}\sqrt{2ry - y^2} + \frac{4}{3}r\int\frac{y^{\frac{1}{2}}dy}{\sqrt{2ry - y^2}}\right].$$

But,

$$\int\frac{y^{\frac{1}{2}}dy}{\sqrt{2ry - y^2}} = \int\frac{dy}{\sqrt{2r - y}} = \int dy(2r - y)^{-\frac{1}{2}} = -2(2r - y)^{\frac{1}{2}};$$

hence,

$$s = 2\pi\sqrt{2r}\left[-\frac{2}{3}y^{\frac{1}{2}}\sqrt{2ry-y^2}-\frac{8}{3}r(2r-y)^{\frac{1}{2}}\right]+C.$$

If we estimate the surface from the plane passing through the centre, we have $C = 0$, since at this point $s = 0$, and $y = 2r$. If we then integrate between the limits $y = 2r$, and $y = 0$, we have,

$$s = \frac{1}{2}\text{ surface } = -\frac{32}{3}\pi r^2; \text{ hence,}$$

$$s = \text{surface} = -\frac{64}{3}\pi r^2,$$

that is, the surface described by the cycloid, when it is revolved around the base, is equal to 64 thirds of the generating circle.

The minus sign should appear before the integral, since the surface is a decreasing function of the variable y (Art. **19**).

Volume generated by the area of the Cycloid.

116. If a cycloid be revolved about its base, it is required to find the measure of the volume which the area will generate.

The differential equation of the cycloid is,

$$dx = \frac{ydy}{\sqrt{2r-y^2}}.$$

If we denote the volume by V, we have (Art. **69**),

$$dV = \frac{\pi y^3 dy}{\sqrt{2ry-y^2}}.$$

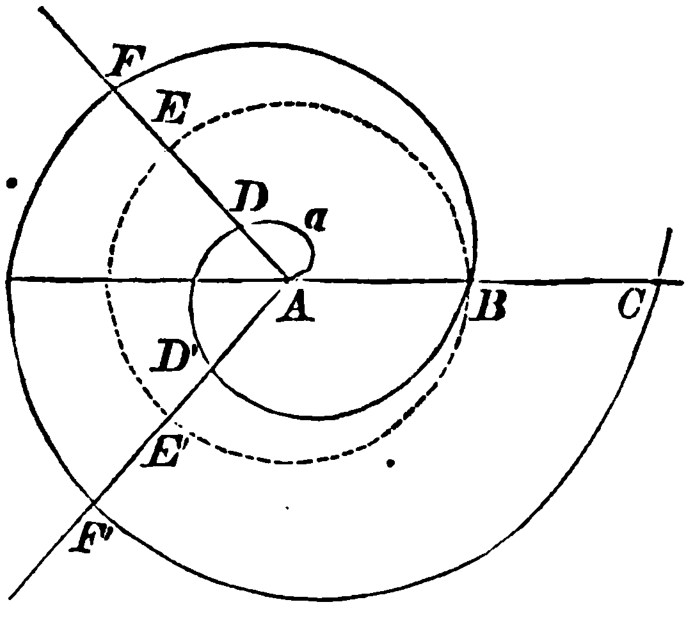
If we apply Formula (E) (Art. **170**), we shall find, after three reductions, that the integral will depend on that of

$$\frac{dy}{\sqrt{2ry - y^2}}.$$

But the integral of this expression is the arc whose versed sine is $\frac{y}{r}$ (Art. **99**). Making the substitutions and reductions, we find the volume equal to five-eighths of the circumscribing cylinder.

Spirals.

117. A *Spiral*, or *Polar Line*, is a curve described by a point which moves along a right line, according to any law whatever, the line having at the same time a uniform angular motion.



Let ABC be a straight line which is to be turned uniformly around the point A. When the motion of the line begins, let us suppose a point to move from A along the line, in the direction ABC. When the line takes the position ADE, the point will have moved along it, to some point, as D, and will have described the arc AaD, of the spiral. When the line takes the position $AD'E'$, the point will have described the curve $AaDD'$, and when the line shall have completed an entire revolution, the point will have described the curve $AaDD'B$.

If the revolutions of the radius-vector be continued, the

generating point will describe an indefinite spiral. The point A, about which the right line revolves, is called the *pole;* the distances AD, AD', AB, are called *radius-vectors* or *radii-vectores ;* and the parts $AaDD'B$, $BFF'C$, described in each revolution, are called *spires.*

If, with the pole as a centre, and AB, the distance passed over by the generating point in the direction of the radius-vector, during the first revolution, as a radius, we describe the circumference BEE', the angular motion of the radius-vector about the pole A, may be measured by the arcs of this circle, estimated from B.

If we designate the radius-vector by u, and the measuring arc, estimated from B, by t, the relation between u and t, may be expressed by the equation,

$$u = at^n,$$

in which n depends on the *law* according to which the generating point moves along the radius-vector, and a on the relation which exists between a *given* value of u, and the corresponding value of t.

General Properties.

118. When n is positive, the spirals represented by the equation,

$$u = at^n,$$

will pass through the pole A. For, if we make $t = 0$, we shall have, $u = 0$.

But if n is negative, the equation will become,

$$u = at^{-n}; \quad \text{or,} \quad u = \frac{a}{t^n},$$

from which we shall have,

for, $\qquad t = 0, \qquad u = \infty,$

and for, $\qquad t = \infty, \qquad u = 0;$

hence, in this class of spirals, the first position of the generating point is at an *infinite distance* from the pole: the point will then approach the pole as the radius-vector revolves, and will only reach it after an *infinite number* of revolutions.

Spiral of Archimedes.

119. If we make $n = 1$, the equation of the spiral becomes,

$$u = at.$$

If we designate two different radii-vectores by u' and u'', and the corresponding arcs by t' and t'', we shall have,

$$u' = at', \quad \text{and} \quad u'' = at'',$$

and consequently,

$$u' : u'' :: t' : t''; \text{ that is,}$$

The radii-vectores are proportional to the measuring arcs, estimated from the initial point.

This spiral is called the spiral of Archimedes.

If we denote by 1, the distance which the generating point moves along the radius-vector, during one revolution, the equation,

$$u = at,$$

will become,

$$1 = at; \quad \text{or,} \quad 1 \times \frac{1}{a} = t.$$

But since t is the circumference of a circle whose radius is 1, we shall have,

$$\frac{1}{a} = 2\pi, \quad \text{and consequently,} \quad a = \frac{1}{2\pi}.$$

Parabolic Spiral.

120. If we make $n = \frac{1}{2}$, and $a = \sqrt{2p}$, we have, for the general equation,

$$u = \sqrt{2p} \times t^{\frac{1}{2}}; \quad \text{or,} \quad u^2 = 2pt,$$

which is the equation of the parabolic spiral.

If $t = 0$, $u = 0$; hence, this spiral passes through the pole.

Hyperbolic Spiral.

121. If we make $n = -1$, the general equation of spirals becomes,

$$u = at^{-1}; \quad \text{or,} \quad ut = a.$$

This spiral is called the *hyperbolic spiral*, because of the analogy which its equation bears to that of the hyperbola, when referred to its asymptotes.

If, in this equation, we make, successively,

$$t = 1, \quad t = \frac{1}{2}, \quad t = \frac{1}{3}, \quad t = \frac{1}{4}, \quad \&c.,$$

we shall have the corresponding values,

$$u = a, \quad u = 2a, \quad u = 3a, \quad u = 4a, \quad \&c.$$

Logarithmic Spiral.

122. Since the relation between u and t is entirely arbitrary, we may, if we please, make,

$$t = \log u.$$

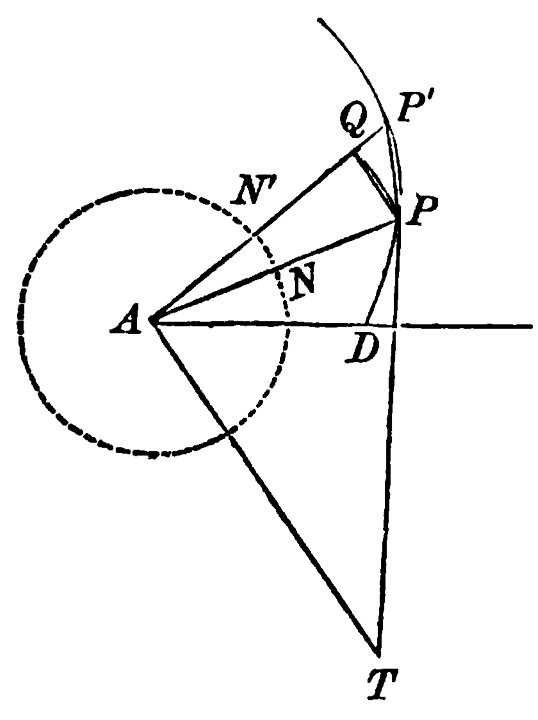
The spiral described by the extremity of the radius-vector, under this supposition, is called the *logarithmic spiral.*

Direction of the measuring arc.

123. The arc, which measures the angular motion of the radius-vector, has been estimated from right to left, and the value of t regarded as positive. If we revolve the radius-vector in a contrary direction, the measuring arc will be estimated from left to right, the sign of t will be changed to negative, and a similar spiral will be described.

Sub-tangent in Polar Curves.

124. The SUB-TANGENT, in spirals, or in *any curve*, referred to polar co-ordinates, is the projection of the tangent on a line drawn through the pole, and perpendicular to the radius-vector passing through the point of contact.



Let A be the pole, $AN = 1$, the radius of the measuring arc, P any point of the curve, TP a tangent at P, and AT,

perpendicular to AP, the sub-tangent. Let AP' be a radius-vector, consecutive with AP, and PQ, an arc described from the centre A.

Then, $NN' = dt$, and $QP' = du$, and, since PQ is parallel to NN', we have, $PQ = udt$. But the arc PQ coincides with its chord (Art. **43**), and since Q is a right angle, the triangles PQP' and TAP are similar; hence,

$$AT : AP :: PQ : QP'; \quad \text{therefore,}$$

Sub-tangent $\quad AT : u :: udt : du.$

Whence, $\quad$ Sub-tangent $AT = \dfrac{u^2dt}{du}, = \dfrac{a}{n}t^{n+1}.$

125. In the spiral of Archimedes, we have,

$$n = 1, \quad \text{and} \quad a = \frac{1}{2\pi};$$

hence,
$$AT = \frac{t^2}{2\pi}.$$

If we make $t = 2\pi$, circumference of the measuring circle, we shall have,

$$AT = 2\pi, \text{ circumference of the measuring circle.}$$

After m revolutions, we shall have

$$t = 2m\pi,$$
and consequently,

$$AT = 2m^2\pi = m.2m\pi; \text{ that is,}$$

The sub-tangent, after m revolutions, is equal to m times

the circumference of the circle whose radius is the radius-vector. This property was discovered by Archimedes.

126. In the hyperbolic spiral, $n = -1$, and the value of the sub-tangent becomes

$$AT = -a; \text{ that is,}$$

The sub-tangent is constant in the hyperbolic spiral.

Angle of the Tangent and Radius-Vector.

127. We see that,

$$\frac{AT}{AP} = \frac{udt}{du},$$

denotes the tangent of the angle which the tangent line makes with the radius-vector.

In the logarithmic spiral, of which the equation is

$$t = \log u,$$

we have, $$dt = \frac{du}{u}M;$$

hence, $$\frac{AT}{AP} = \frac{udt}{du} = M; \text{ that is,}$$

In the logarithmic spiral, the angle formed by the tangent and the radius-vector passing through the point of contact, is constant; and the tangent of the angle is equal to the modulus of the system of logarithms.

If t is the Naperian logarithm of u, M is 1 (Art. **89**), and the angle will be equal to 45°.

Value of the Tangent.

128. The value of the tangent, in a curve referred to polar co-ordinates, is,

$$PT = \sqrt{\overline{AP}^2 + \overline{AT}^2} = u\sqrt{1 + \frac{u^2dt^2}{du^2}}.$$

Differential of the Arc.

129. To find the differential of the arc, which we denote by z, we have,

$$PP' = \sqrt{\overline{QP'}^2 + \overline{QP}^2};$$

or, by substituting for PP', QP', and PQ, their values, when P and P' are consecutive, we have,

$$dz = \sqrt{du^2 + u^2dt^2}.$$

Differential of the Area.

130. The differential of the area ADP, when referred to polar co-ordinates, is not an elementary rectangle, as when referred to rectangular axes, but is the elementary sector APP'. The area of this triangle is equal to $\frac{AP' \times PQ}{2}$. If we denote the differential by ds, we have,

$$ds = \frac{AP' \times QP}{2} = \frac{(u + du)udt}{2};$$

or, omitting the infinitely small quantity of the second order, $ududt$ (Art. **20**),

$$ds = \frac{u^2dt}{2},$$

which is the differential of the area of any segment of a polar line.

Areas of Spirals.

131. If we denote by s, the area described by the radius-vector, we have (Art. **130**),

$$ds = \frac{u^2 dt}{2};$$

and placing for u its value, at^n (Art. **117**),

$$ds = \frac{a^2 t^{2n} dt}{2}, \quad \text{and} \quad s = \frac{a^2 t^{2n+1}}{4n+2} + C.$$

If n is positive, $C = 0$, since the area is 0, when $t = 0$. After one revolution of the radius-vector, $t = 2\pi$, and we have,

$$s = \frac{a^2 (2\pi)^{2n+1}}{4n+2},$$

which is the area included within the first spire.

132. In the spiral of Archimedes, (Art. **119**),

$$a = \frac{1}{2\pi}, \quad \text{and} \quad n = 1;$$

hence, for this spiral we have,

$$s = \frac{t^3}{24\pi^2},$$

which becomes $\frac{\pi}{3}$, after one revolution of the radius-vector; the unit of the number $\frac{\pi}{3}$, being a square whose side is 1. Hence, *the area included by the first spire, is equal to one-third of the area of the circle whose radius is the radius-vector, after the first revolution.*

In the second revolution, the radius-vector describes a second time, the area described in the first revolution; and in any succeeding revolution, it will pass over, or re-describe, all the area before generated. Hence, to find the area, at the end of the mth revolution, we must integrate between the limits,

$$t = (m - 1)2\pi, \qquad \text{and} \qquad t = m.2\pi,$$

which gives,

$$s = \frac{m^3 - (m-1)^3}{3}\pi.$$

If it be required to find the area between any two spires, as between the mth and the $(m + 1)$th, we have for the whole area to the $(m + 1)$th spire,

$$\frac{(m+1)^3 - m^3}{3}\pi;$$

and subtracting the area to the mth spire, gives,

$$s = \frac{(m+1)^3 - 2m^3 + (m-1)^3}{3}\pi = 2m\pi,$$

for the area between the mth and $(m + 1)$th spires.

If we make $m = 1$, we shall have the area between the first and second spires equal to 2π; hence, *the area between the mth and $(m + 1)$th spires, is equal to m times the area between the first and second.*

133. In the hyperbolic spiral, $n = -1$, and we have,

$$ds = \frac{a^2t^{-2}}{2}dt, \qquad \text{and} \qquad s = -\frac{a^2}{2t}.$$

The area s will be infinite, when $t = 0$, but we can

find the area included between any two radii-vectores b and c, by integrating between the limits $t = b$ and $t = c$, which will give,

$$s = \frac{a^2}{2}\left(\frac{1}{b} - \frac{1}{c}\right).$$

134. In the logarithmic spiral, $t = lu$; hence,

$$dt = \frac{du}{u}, \qquad \text{and,} \quad \frac{u^2dt}{2} = \frac{udu}{2};$$

hence, $$s = \int \frac{udu}{2} = \frac{u^2}{4} + C;$$

and by considering the area $s = 0$, when $u = 0$, we have $C = 0$, and

$$s = \frac{u^2}{4}.$$

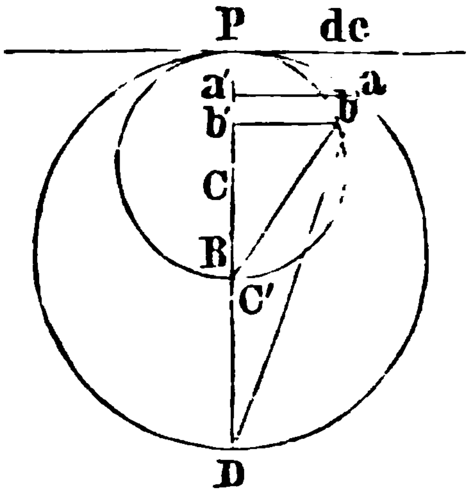
CURVATURE.

135. THE CURVATURE of a plane curve, at any point, is its *tendency* to depart from the tangent drawn to the curve at that point. This tendency is measured by the distance which a point, moving on the curve, departs from the tangent in passing over a unit of length, denoted by the differential of the arc. In the same circle, or in equal circles, the tendency to depart from a tangent, at any point, is always the same; hence, the curvature of a circle, at all points, is constant.

Curvature of a circle is inversely as the radius.

136. Let C and C' be the centres of two unequal circles, having a common tangent at P. If we suppose the

arcs to be the independent variables, we can denote the differential of one arc by Pb, and the differential of the other, by an equal arc Pa. Then, having drawn bB and aD, and the sines, bb', aa', and recollecting that each arc is equal to its corresponding chord, (Art. 43), we have, by denoting the radii by r and r',*



$$\overline{Pb}^2 = 2r.Pb', \qquad \text{and} \qquad \overline{Pa}^2 = 2r'.Pa';$$

since the arcs are equal, and $Pb' = db$, and $Pa' = ca$,

$$2r.db = 2r'.ca; \quad \text{hence,}$$

$$db \; : \; ac \; :: \; \frac{1}{r} \; : \; \frac{1}{r'}; \quad \text{that is,}$$

The curvature of a circle varies inversely as its radius; hence, *the reciprocal of the radius of a circle may be assumed as the measure of its curvature.*

Orders of Contact.

137. If two plane curves have one point in common, there is one set of co-ordinates (which may be denoted by x'', y''), that will satisfy the equations of both curves. If the curves have a second point in common, *consecutive with the first*, they will have a common tangent, at the common point, and the first differential coefficients will also be equal (Art. 43); this is called, *a contact of the first order*, If the curves have a third point in common, consecutive with the second, the second differential coefficients will be

* Legendre, Bk. IV. P. 23.

equal (Art. 73); this is called, *a contact of the second order.*

Generally, two curves have a contact of the nth order, when they have a common point, and the first n successive differential coefficients of the common ordinate, equal to each other.

Osculatory Curves.

138. AN OSCULATRIX, is a curve which has a higher order of contact with a given curve, at a given point, than any other curve of the same kind. The osculatory circle is by far the most important of all the osculatrices; for it is this circle which measures the curvature of all plane curves.

Osculatory Circle.

139. The general equation of a circle, referred to rectangular co-ordinates (Bk. II., Art. 5), is,

$$(x - \alpha)^2 + (y - \beta)^2 = R^2 \quad . \quad . \quad . \quad (1.)$$

in which α and β are the co-ordinates of the centre, and x and y the co-ordinates of any point of the curve.

If we regard α, β, and R, as constants, and differentiate the equation twice, and then find the differential coefficients of the first and second order, we have,

$$\frac{dy}{dx} = -\frac{x - \alpha}{y - \beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

and,
$$\frac{d^2y}{dx^2} = -\frac{1 + \frac{dy^2}{dx^2}}{y - \beta} \quad . \quad . \quad . \quad . \quad (3.)$$

In Equation (1) there are three arbitrary constants, α, β, and R; and values may be assigned to these, at pleasure, so as to cause the circle to fullfil three conditions, and three only.

If we have any plane curve whose equation is of the form,

$$y = f(x);$$

and find, from this equation, the first and second differential coefficients, for any point whose co-ordinates are x'', y'', we may then attribute such values to α, β, and R, as shall make,

$$x = x'', \qquad y = y''; \qquad \text{also,}$$

$$\frac{dy}{dx} = \frac{dx''}{dy''}, \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d^2y''}{dx''^2}.$$

As no further general relations can be established between the differential coefficients of the circle and curve, this circle will be osculatory to the curve at the point whose co-ordinates are x'', y'' (Art. **138**). Since the co-ordinates of a point, and the differential coefficients of the first and second order, determine three consecutive points (Art. **73**), it follows that, *the osculatory circle passes through three consecutive points of the curve, at the point of osculation.*

Limit of the Orders of Contact.

140. It is seen that the highest order of contact which a circle can have with any curve, is denoted by the number of arbitrary constants which enters into its equation, less 1; and the same is true for any other osculatrix.

Although it is impossible to *assign* a higher order of contact, to a circle, than the *second*, yet, at the vertices of the transverse and conjugate axes of the conic sections, the conditions which make the circle osculatory, also make the third differential coefficient zero, and hence give a contact of the third order. In general, when the order of contact is *even*, and the curve symmetrical with the normal at the point of osculation, the conditions imposed will give a contact of the next higher order.

Radius of Curvature.

141. If we find the value of R from Equations (1), (2), and (3), we have,

$$R = \pm \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y} \quad . \quad . \quad . \quad . \quad (4.)$$

which is the general value for the radius of the osculatory circle.

If we denote the arc by z, we have (Art. **52**),

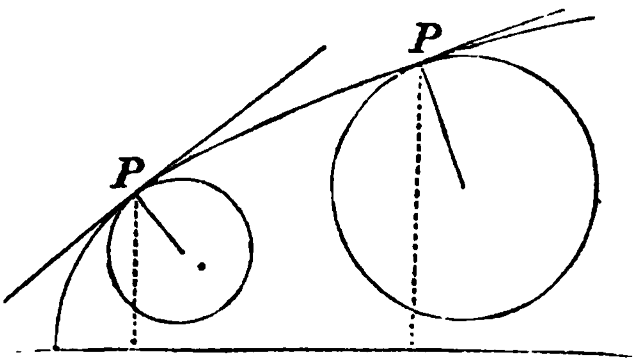
$$dz = \sqrt{dx + dy};$$

whence,

$$R = \pm \frac{dz^3}{dxd^2y} \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

Measure of Curvature.

142. The curvature of a curve, at any point, is measured by the curvature of the osculatory circle at that point; hence, it is the reciprocal of the radius (Art. **136**).



If we assume two points, P and P', either on the

same, or on different curves, and find the radii r and r' of the circles which are osculatory at these points, then,

$$\text{curvature at } P : \text{curvature at } P' :: \frac{1}{r} : \frac{1}{r'}.$$

143. To find the radius of curvature, at any point of a plane curve, whose equation is of the form,

$$y = f(x).$$

Differentiate the equation twice, and substitute the values of the first and second differentials in Equation (4); the resulting equation will indicate the value of R for that point.

If we use the $+$ sign, when the curve is convex toward the axis of abscissas, and the $-$ sign when it is concave, the essential sign of R will be positive, when R is an increasing function of x.

Radius of Curvature for Lines of the Second Order.

144. The general equation for lines of the second order (Bk. V, Art. **42**), is,

$$y^2 = mx + nx^2,$$

which gives, by differentiation,

$$dy = \frac{(m + 2nx)dx}{2y}, \quad dx^2 + dy^2 = \frac{[4y^2 + (m + 2nx)^2]dx^2}{4y^2},$$

$$d^2y = \frac{2nydx^2 - (m + 2nx)dxdy}{2y^2} = \frac{[4ny^2 - (m + 2nx)^2]dx^2}{4y^3}$$

Substituting these values in the equation,

$$R = -\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y},$$

we obtain, $$R = \frac{[4(mx + nx^2) + (m + 2nx)^2]^{\frac{3}{2}}}{2m^2};$$

which is the radius of curvature in lines of the second order, for any abscissa x.

145. If we make $x = 0$, we have,

$$R = \frac{1}{2}m = \frac{B^2}{A};$$

that is, in lines of the second order, *the radius of curvature at the vertex of the transverse axis is equal to half the parameter of that axis.*

146. If it is required to find the value of the radius of curvature at the vertex of the conjugate axis of an ellipse, we make (Bk. V, Art. **42**),

$$m = \frac{2B^2}{A}, \quad n = -\frac{B^2}{A^2}, \quad \text{and} \quad x = A,$$

which gives, after reducing,

$$R = \frac{A^2}{B}; \text{ hence,}$$

The radius of curvature at the vertex of the conjugate axis of an ellipse is equal to half the parameter of that axis.

147. In the case of the parabola, in which $n = 0$, the general value of the radius of curvature becomes,

$$R = \frac{(m^2 + 4mx)^{\frac{3}{2}}}{2m^2}.$$

If we make $x = 0$, we shall have the radius of curvature at the vertex, equal to $\frac{m}{2}$, or *one-half the parameter.*

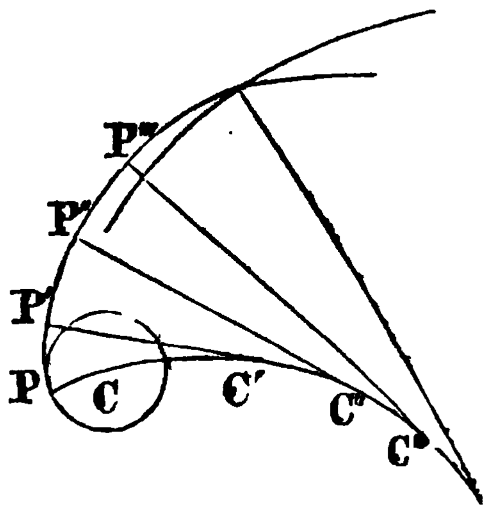
148. If we compare the value of the radius of curvature (Art. **144**), with that of the normal line found in Art. **49**, we shall have,

$$R = \frac{(\text{normal})^3}{\frac{1}{4}m^2}; \text{ that is,}$$

The radius of curvature, at any point, is equal to the cube of the normal divided by half the parameter squared; and hence, *the radii of curvature, at different points of the same curve, are to each other as the cubes of the corresponding normals; and the curvature is proportional to the reciprocals of those cubes.*

Evolute Curves.

149. An Evolute curve is the locus of the centres of all the circles which are osculatory to a given curve. The given curve is called the Involute.



If at different points, P, P', P'', &c., of an involute, or given curve, normals, PC, $P'C'$, &c., be drawn, and distances laid off on them, on the concave side of the arc, each equal to the radius of curvature at the point, then the curve drawn through the extremities C, C', C'', &c., of these radii of curvature, is the evolute curve.

A normal to the Involute is tangent to the Evolute.

150. Resuming the consideration of the first three equations of Art. **139**, and changing slightly the forms of (2) and (3), we have,

$$(x - \alpha)^2 + (y - \beta)^2 = R^2 \quad . \quad . \quad . \quad (1.)$$

$$(x - \alpha)dx + (y - \beta)dy = 0 \quad . \quad . \quad . \quad (2.)$$

$$dx^2 + dy^2 + (y - \beta)d^2y = 0 \quad . \quad . \quad . \quad (3.)$$

Equations (2) and (3) were derived from Equation (1), under the supposition that α, β, and R, were arbitrary constants, and of such values as to cause the circle to be osculatory to a given curve, at a given point.

If now, we suppose the osculatory circle to move along the involute, continuing osculatory to it, the five quantities, R, α, β, y, dy, will all be functions of the independent variable x, and α and β will be the co-ordinates of the evolute curve.

If we differentiate Equations (1) and (2) under this hypothesis, we have,

$$(x - \alpha)dx + (y - \beta)dy - (x - \alpha)d\alpha - (y - \beta)d\beta = RdR,$$

$$dx^2 + dy^2 + (y - \beta)d^2y - d\alpha dx - d\beta dy = 0.$$

Combining the first with Equation (2), and the second with (3), we obtain,

$$- (y - \beta)\, d\beta - (x - \alpha)d\alpha = RdR, \quad . \quad (4.)$$

$$- d\alpha dx - d\beta dy = 0 \quad . \quad . \quad . \quad (5.)$$

From the last equation we have,

$$\frac{d\beta}{d\alpha} = -\frac{dx}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

But Equation (2) may be placed under the form,

$$y - \beta = -\frac{dx}{dy}(x - \alpha), \quad \text{or,} \quad \beta - y = -\frac{dx}{dy}(\alpha - x) \quad . \quad (7.)$$

Substituting for $-\frac{dx}{dy}$, its value $\frac{d\beta}{dx}$, we have,

$$y - \beta = \frac{d\beta}{dx}(x - \alpha) \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Since Equations (7) and (8) are the same under different forms, they represent one and the same line.

Equation (7) is the equation of a normal to the involute at a point whose co-ordinates are x and y, and passes through any point whose co-ordinates are α and β (Art. 44). Equation (8) is the equation of a tangent to the evolute at a point whose co-ordinates are α and β, and passes through any point whose co-ordinates are x and y (Art. 43); therefore,

The radius of curvature which is normal to the involute is tangent to the evolute.

Evolute and radius of curvature increase or decrease by the same quantity.

151. Combining Equations (2) and (6), we have,

$$x - \alpha = \frac{d\alpha}{d\beta}(y - \beta) \quad . \quad . \quad . \quad . \quad (9.)$$

Substituting this value of $x - \alpha$ in Equation (1), we have, after reduction,

$$(y - \beta)^2 \left(\frac{d\alpha^2 + d\beta^2}{d\beta^2}\right) = R^2 \quad . \quad . \quad (10.)$$

Substituting the same value in Equation (4), reducing, and squaring both members, we obtain,

$$(y - \beta)^2 \frac{(d\alpha^2 + d\beta^2)^2}{d\beta^2} = R^2(dR)^2 \quad . \quad (11.)$$

Dividing (11) by (10), member by member, and taking the root,

$$\sqrt{d\alpha^2 + d\beta^2} = dR \quad . \quad . \quad . \quad . \quad (12.)$$

But since α and β are the co-ordinates of the evolute, if we denote this curve by z, we shall have (Art. **52**),

$$dR = dz, \qquad dR - dz = 0, \qquad d(R - z) = 0;$$

whence, $\qquad R - z = \text{a constant (Art. 17)};$

which, if we denote by c, gives,

$$R = z + c \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

Since the difference between R and z is constant, it follows that any change in one, will produce a *corresponding* and *equal change* in the other.

If we draw any two radii of curvature, as PC, $P'C'$, and denote them by R and R', and the corresponding arcs of the evolute by z and z', we have,

$$R = z + c, \quad \text{and} \quad R' = z' + c;$$

whence, $\qquad R' - R = z' - z;$ that is,

The difference between any two radii of curvature is

equal to the arc of the evolute intercepted between their extremities.

If we make $z = 0$, and denote the corresponding value of R by r, we have,

$$r = 0 + c = c; \text{ hence,}$$

The constant c, is equal to the radius of curvature passing through the origin of arc of the evolute.

If we suppose C to be the origin of arc of the evolute, then, $CP = r = c$; and any radius of curvature, as $C''P'$, will be equal in length to the line $C'CP$. If then the evolute be developed, or unrolled, as it were, about the movable centre of the osculatory circle, the other extremity of the radius of curvature will describe the involute curve.

Evolute of the Cycloid.

152. Let us resume the equation for the radius of curvature (Art. **141**),

$$R = -\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y} \quad . \quad . \quad . \quad . \quad (1.)$$

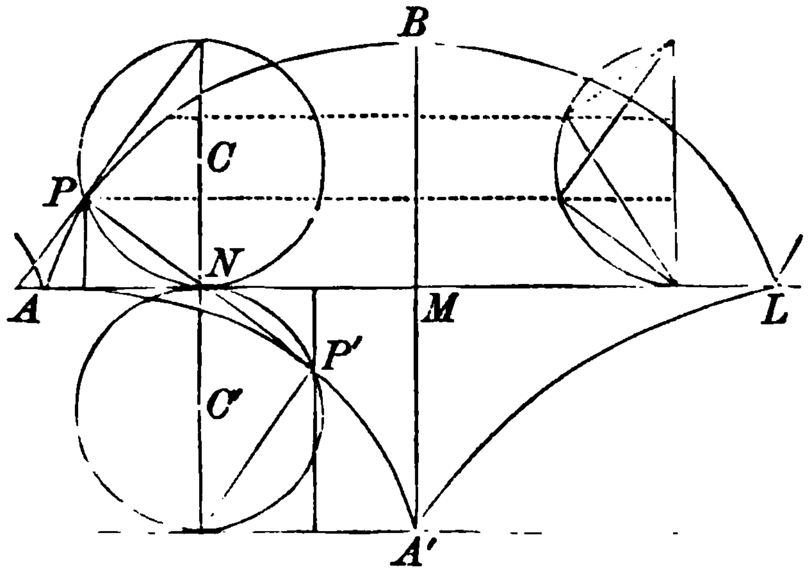
If, in this equation, we substitute the value of d^2y found in Art. **113**, we have,

$$R = 2^{\frac{3}{2}}(ry)^{\frac{1}{2}} = 2\sqrt{2ry} \quad . \quad . \quad . \quad . \quad (2.)$$

hence (Art. **111**), *The radius of curvature is double the normal;* therefore, when the generating circle moves from A towards M, any radius of curvature, as PP', will be double the normal PN.

If, in Equation (2), we make $y = 0$, we have, $R = 0$;

If we make $y = MB = 2r$, we have, $R = 4r$.



That is, the radius of curvature is zero at the point A, and twice the diameter of the generating circle at B.

Since the radius of curvature and evolute are both zero at the point A, and since they increase equally (Art. **151**), it follows that the length of the evolute $AP'A'$ is equal to $A'MB$, or *twice the diameter of the generating circle.*

When the point of contact, N, shall have reached M, the point P, will have described the involute APB, and the point P', the evolute $AP'A'$. If we describe a circle on $A'M = \frac{1}{2}A'B$, it will be equal to the generating circle of the cycloid, and the two circles will touch each other at M. Draw $A'X$ parallel to AL.

If now we suppose the circle whose centre is C, to roll along the base from M to A, and the circle whose centre is C', to roll from A' to X, keeping the centres C and C', in a line perpendicular to the base AL, the point P, of the upper circle, will re-describe the involute BPA, and the point P', will re-describe the evolute $A'P'A$. But since the generating circles are equal, and since they are rolled over equal bases, the curves generated will be equal; hence, *the involute and evolute are equal curves.*

The part of the involute beginning at A, is identical with the part of the evolute beginning at A'.

Since the involute and evolute are equal, the length of the involute APB, is equal to twice the diameter of the generating circle; or the length of the entire cycloidal arc $APBL$, is equal to the entire evolute $AP'A'L$, or to four times the diameter of the generating circle.

Equation of the Evolute.

153. The equation of the evolute may be readily found by combining the equations,

$$y - \beta = -\frac{dx^2 + dy^2}{d^2y}, \qquad x - \alpha = \frac{dy(dx^2 + dy^2)}{dxd^2y},$$

with the equation of the involute curve.

1st. Find, from the equation of the involute, the values of

$$\frac{dy}{dx} \quad \text{and} \quad d^2y,$$

and substitute them in the last two equations; there will result two new equations, involving α, β, x, and y.

2d. Combine these equations with the equation of the involute, and eliminate x and y; the resulting equation will contain α, β, and constants, and will be the equation of the evolute curve.

Evolute of the common Parabola.

154. Let us take, as an example, the common parabola, of which the equation is,

$$y^2 = mx.$$

We shall then have,

$$\frac{dy}{dx} = \frac{m}{2y}, \qquad d^2y = -\frac{m^2dx^2}{4y^3};$$

and hence,

$$y - \beta = \frac{4y^3}{m^2}\left(\frac{4y^2 + m^2}{4y^2}\right) = \frac{4y^3 + m^2y}{m^2} = \frac{4y^3}{m^2} + y;$$

and observing that the value of $x - \alpha$ is equal to that of $y - \beta$ multiplied by $-\frac{dy}{dx}$, we have,

$$x - \alpha = -\frac{4y^2 + m^2}{2m};$$

hence we have,

$$-\beta = \frac{4y^3}{m^2}, \qquad \text{and} \qquad x - \alpha = -\frac{2y^2}{m} - \frac{m}{2};$$

substituting for y its value in the equation of the involute,

$$y = m^{\frac{1}{2}}x^{\frac{1}{2}},$$

we obtain,

$$-\beta = \frac{4x^{\frac{3}{2}}}{m^{\frac{1}{2}}}; \qquad x - \alpha = -2x - \frac{m}{2};$$

and by eliminating x, we have,

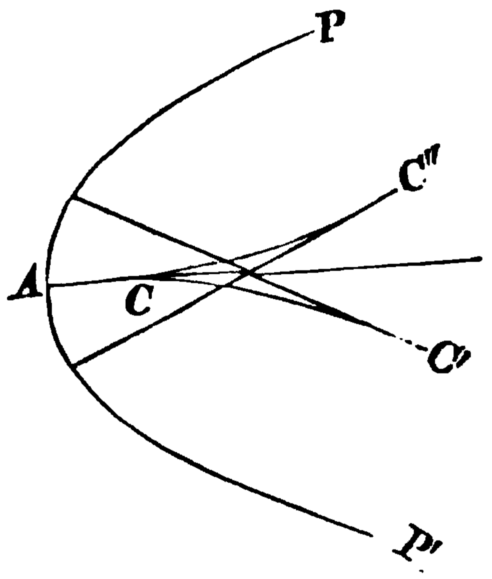
$$\beta^2 = \frac{16}{27m}\left(\alpha - \frac{1}{2}m\right)^3,$$

which is the equation of the evolute.

If we make $\beta = 0$, we have,

$$\alpha = \frac{1}{2}m;$$

and hence, the evolute meets the axis of abscissas at a distance from the origin equal to half the parameter. If the origin of co-ordinates be transferred from A to this point, we shall have,



$$\alpha' = \alpha - \frac{1}{2}m,$$

and consequently,

$$\beta^2 = \frac{16}{27m}\alpha'^3.$$

The equation of the curve shows that it is symmetrical with respect to the axis of abscissas, and that it does not extend in the direction of the negative values of α'. The evolute CC' corresponds to the part AP of the involute, and CC'' to the part AP'. Both are convex towards the axis of X.

INTEGRAL CALCULUS.

Nature of Integration.

155. In the Differential Calculus, we have developed a system of principles, and given a series of rules, by means of which we deduce, from any given function, two others; the first of which is called the Differential coefficient, and the second, the Differential (Art. **25**). In the Integral Calculus, we have to return from the differential, to the function from which it was derived.

This operation, as a fundamental problem, involves the summation of a series of an infinite number of terms, each of which is infinitely small (Art. **56**). No general rule for the summation of such a series has yet been discovered; and hence, we are obliged to resort, in each particular case, to the operation of reducing the given differential to some equivalent one, whose integral is known (Art. **34**).

Forms of differentials having known Algebraic Functions.

156. We have found (Art. **35**), that every differential monomial of the form,

$$Ax^m dx,$$

in which m is any real number, except -1, may be immediately integrated; and when $m = -1$, the differential becomes that of a logarithmic function, and its integral is $A\,lx$ (Art. **89**).

157. We have seen that every differential binomial of the form,

$$(a + bx^n)^m x^{n-1} dx,$$

in which the exponent of the variable without the parenthesis is less by 1 than the exponent of the variable within, can be immediately integrated (Art. **41**).

158. We have seen that every function of the form,

$$Xdx,$$

in which X can be developed into a series in terms of the ascending powers of x, has an approximate integral which may be readily found (Art. **42**).

Forms of differentials having known Logarithmic Functions.

159. Any function of the form,

$$A\frac{dx}{x},$$

in which the numerator is the differential of the denominator, can be immediately integrated, since the integral is equal to $A\,lx$ (Art. **89**). In Art. **91**, we have given five other forms of differentials, whose corresponding functions are logarithms.

Forms of differentials having known Circular Functions.

160. In Art. **99**, we have found sixteen differential expressions, each of which has a known integral corresponding to it, and which, being differentiated, will of course produce the given differential.

In all the classes of functions, any differential expression may be considered as integrated, when it is reduced to one of the known forms; and the operations of the Integral Calculus consist, mainly, in making such transformations of given differential expressions, as shall reduce them to equivalent ones, whose integrals are known.

INTEGRATION OF RATIONAL FRACTIONS.

161. Every rational fraction may be written under the form,

$$\frac{Px^{n-1} + Qx^{n-2} \quad . \ . \ . \ . \ + Rx + S}{P'x^n \quad + Q'x^{n-1} \quad . \ . \ . \ + R'x + S'}\,dx,$$

in which the exponent of the highest power of the variable in the numerator is less by 1 than in the denominator. For, if the greatest exponent in the numerator was equal to, or exceeded the greatest exponent in the denominator, a division might be made, giving one or more entire terms for a quotient, and a remainder, in which the exponent of the leading letter would be less by at least 1, than the exponent of the leading letter in the divisor. The entire terms could then be integrated, and there would remain a fraction under the above form.

EXAMPLES.

1. Let it be required to integrate the expression,

$$\frac{2adx}{x^2 - a^2}.$$

By decomposing the denominator into its factors, we have,

$$\frac{adx}{x^2 - a^2} = \frac{2adx}{(x - a)(x + a)}.$$

Let us make,

$$\frac{2adx}{(x - a)(x + a)} = \left(\frac{A}{x - a} + \frac{B}{x + a}\right)dx,$$

in which A and B are constants, whose values may be found by the method of indeterminate co-efficients.* To find these constants, reduce the terms of the second member of the equation to a common denominator; we shall then have,

$$\frac{adx}{(x - a)(x + a)} = \frac{(Ax + Aa + Bx - Ba)dx}{(x - a)(x + a)}.$$

Comparing the two members of the equation, we find,

$$2a = Ax + Aa + Bx - Ba;$$

or, by arranging with reference to x,

$$(A + B)x + (A - B - 2)a = 0; \text{ hence,}$$

$$A + B = 0, \qquad \text{and} \qquad (A - B - 2)a = 0;$$

whence, $\quad A = 1, \quad B = -1.$

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* Bourdon, Art. **194.** University, Art. **180.**

Substituting these values for A and B, we obtain,

$$\frac{2adx}{x^2 - a^2} = \frac{dx}{x - a} - \frac{dx}{x + a};$$

integrating, we find (Art. 89),

$$\int \frac{adx}{x^2 - a^2} = l(x - a) - l(x + a) + C; \text{ consequently,}$$

$$\int \frac{adx}{x^2 - a^2} = l\left(\frac{x - a}{x + a}\right) + C.$$

2. Find the integral of,

$$\frac{3x - 5}{x^2 - 6x + 8}dx.$$

Resolving the denominator into its two binomial factors, $(x - 2)$, and $(x - 4)$, we have,

$$\frac{3x - 5}{x^2 - 6x + 8} = \frac{A}{x - 2} + \frac{B}{x - 4}; \text{ hence,}$$

$$\frac{3x - 5}{x^2 - 6x + 8} = \frac{Ax - 4A + Bx - 2B}{x^2 - 6x + 8};$$

by comparing the coefficients of x, we have,

$$-5 = -4A - 2B, \qquad 3 = A + B,$$

which gives, $\quad B = \frac{7}{2}, \quad A = -\frac{1}{2};$

substituting these values, we have,

$$\int \frac{3x - 5}{x^2 - 6x + 8}dx = -\frac{1}{2}\int \frac{dx}{x - 2} + \frac{7}{2}\int \frac{dx}{x - 4} + C$$

$$= \frac{7}{2}\log(x - 4) - \frac{1}{2}\log(x - 2) + C.$$

Hence, for the integration of rational fractions:

1st. *Resolve the fraction into partial fractions, of which the numerators shall be constants, and the denominators factors of the denominator of the given fraction.*

2d. *Find the values of the numerators of the partial fractions, and multiply each by dx.*

3d. *Integrate each partial fraction separately, and the sum of the integrals thus found will be the integral sought.*

INTEGRATION BY PARTS.

162. The integration of differentials is often effected by resolving them into two parts, of which one has a known integral.

We have seen (Art. **27**), that,

$$d(uv) = udv + vdu,$$

whence, by integrating,

$$uv = \int udv + \int vdu,$$

and, consequently,

$$\int udv = uv - \int vdu.$$

Hence, if we have a differential of the form Xdx, which can be decomposed into two factors P and Qdx, of which one of them, Qdx, can be integrated, we shall have, by making $\int Qdx = v$, and $P = u$,

$$\int Xdx = \int PQdx = \int udv = uv - \int vdu \quad . \quad (1.)$$

in which it is only required to integrate the term $\int vdu$.

EXAMPLES.

1. Integrate the expression, $x^3dx\sqrt{a^2+x^2}$.

This may be divided into the two factors,

$$x^2, \quad \text{and} \quad xdx\sqrt{a^2+x^2},$$

of which the second is integrable (Art. **41**).

Put, $\quad u = x^2, \quad$ and $\quad dv = xdx\sqrt{a^2+x^2};$

then,

$$du = 2xdx, \quad \text{and} \quad v = \int xdx\sqrt{a^2+x^2} = \frac{(a^2+x^2)^{\frac{3}{2}}}{3}.$$

Substituting these values in Formula (1),

$$\int udv = x^2\left(\frac{a^2+x^2}{3}\right)^{\frac{3}{2}} - \int \frac{(a^2+x^2)^{\frac{3}{2}}}{3} \times 2xdx;$$

and finally,

$$\int x^3dx\sqrt{a^2+x^2} = x^2\left(\frac{a^2+x^2}{3}\right)^{\frac{3}{2}} - \frac{2}{15}(a^2+x^2)^{\frac{5}{2}} + C.$$

2. Integrate the expression, $\dfrac{x^2dx}{(a^2-x^2)^{\frac{3}{2}}}$.

The two factors are, $\quad x, \quad$ and $\quad xdx(a^2-x^2)^{-\frac{3}{2}}$.

$$u = x; \qquad dv = xdx(a^2-x^2)^{-\frac{3}{2}}; \qquad v = \frac{1}{\sqrt{(a^2-x^2)}}.$$

$$\int udv = \frac{u}{\sqrt{a^2-x^2}} + \int \frac{dx}{\sqrt{a^2-x^2}}; \quad \text{whence,}$$

$$\int \frac{x^2dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{a^2-x^2}} + \sin^{-1}\frac{x}{a} \quad \text{(Art. 99).}$$

INTEGRATION OF BINOMIAL DIFFERENTIALS.

Form of Binomial.

163. Every binomial differential may be placed under the form,

$$x^{m+1}dx(a + bx^n)^p,$$

in which m and n are whole numbers, and n positive; and in which p is entire or fractional, positive or negative.

1. For, if m and n are fractional, the binomial takes the form,

$$x^{\frac{1}{3}}dx(a + bx^{\frac{1}{2}})^p.$$

If we make $x = z^6$, that is, if we substitute for x, another variable, z, with an exponent equal to the least common multiple of the denominators of the exponents of x, we shall have,

$$x^{\frac{1}{3}}dx(a + bx^{\frac{1}{2}})^p = 6z^7dz(a + bz^3)^p,$$

in which the exponents of the variable are entire.

2. If n is negative, we have,

$$x^{m-1}dx(a + bx^{-n})^p,$$

and by making $x = \frac{1}{z}$, we obtain,

$$-z^{-m-1}dz(a + bz^n)^p,$$

in which n is positive.

3. If x enters into both terms of the binomial, giving the form,

$$x^{m-1}dx(ax^r + bx^n)^p,$$

in which the lowest power of x is written in the first term, we divide the binomial within the parenthesis by x^r, and multiply the factor without by x^{rp}; this gives,

$$x^{m+pr-1}dx(a + bx^{n-r})^p,$$

which is of the required form when the exponent $m + p^{2-1}$, is a whole number, and may easily be reduced to it, when that exponent is fractional.

When a Binomial can be integrated.

164.—1. If p is entire and positive, it is plain that the binomial can be integrated. For, when the binomial is raised to the indicated power, there will be a finite number of terms, each of which, after being multiplied by $x^{m-1}dx$, may be integrated (Art. **35**).

2. If $m = n$, the binomial can be integrated (Art. **41**).

3. If p is entire, and negative, the binomial will take the form,

$$\frac{x^{m-1}dx}{(a + bx^n)^p};$$

which is a rational fraction.

Formula A.

For diminishing the exponent of the variable without the parenthesis.

165. Let us resume the differential binomial,

$$x^{m-1}dx(a + bx^n)^p.$$

If we multiply by the two factors, x^n and x^{-n}, the value will not be changed, and we obtain,

$$x^{m-n}x^{n-1}dx(a+bx^n)^p.$$

Now, the factor $x^{n-1}dx(a+bx^n)^p$ is integrable, whatever be the value of p (Art. 41). Denoting the first factor, x^{m-n} by u, and the second by dv, we have,

$$du = (m-n)x^{m-n-1}dx, \qquad \text{and} \qquad v = \frac{(a+bx^n)^{p+1}}{(p+1)nb};$$

and, consequently,
$$\int x^{m-1}dx(a+bx^n)^p =$$

$$\frac{x^{m-n}(a+bx^n)^{p+1}}{(p+1)nb} - \frac{m-n}{(p+1)nb}\int x^{m-n-1}dx(a+bx^n)^{p+1}.$$

But,
$$\int x^{m-n-1}\,dx(a+bx^n)^{p+1} =$$

$$\int x^{m-n-1}\,dx(a+bx^n)^p\,(a+bx^n) =$$

$$a\int x^{m-n-1}\,dx(a+bx^n)^p + b\int x^{m-1}\,dx(a+bx^n)^p;$$

substituting this last value in the preceding equation, and collecting the terms containing,

$$\int x^{m-1}\,dx(a+bx^n)^p,$$

we have,
$$\left(1+\frac{m-n}{(p+1)n}\right)\int x^{m-1}\,dx(a+bx^n)^p =$$

$$\frac{x^{m-n}\,(a+bx^n)^{p+1} - a(m-n)\int x^{m-n-1}\,dx(a+bx^n)^p}{(p+1)nb};$$

whence,

$$(\mathbf{A}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \int x^{m-1} dx(a + bx^n)^p =$$

$$\frac{x^{m-n}(a + bx^n)^{p+1} - a(m - n)\int x^{m-n-1} dx(a + bx^n)^p}{b(pn + m)}.$$

This formula reduces the differential binomial,

$$\int x^{m-1} dx(a + bx^n)^p, \qquad \text{to} \qquad \int x^{m-n-1} dx(a + bx^n)^p;$$

and by a similar operation, we should find,

$$\int x^{m-n-1} dx(a + bx^n)^p, \text{ to depend on, } \int x^{m-2n-1} dx(a + bx^n)^p;$$

consequently, *each operation diminishes the exponent of the variable without the parenthesis by the exponent of the variable within.*

After the second integration, the factor $m - n$, of the second term, becomes $m - 2n$; and after the third, $m - 3n$, &c. If m is a multiple of n, the factor $m - n$, $m - 2n$, $m - 3n$, &c., will finally become equal to 0, and then the differential into which it is multiplied will disappear, and the given differential can be integrated. Hence, *a differential binomial can be integrated, when the exponent of the variable without the parenthesis plus* 1, *is a multiple of the exponent within.*

APPLICATIONS.

166. We have frequent occasion to integrate differential binomials of the form,

$$\frac{x^m dx}{\sqrt{a^2 - x^2}} = x^m dx(a^2 - x^2)^{-\frac{1}{2}}.$$

The differential binomial $x^{m-1}\,dx(a + bx^n)^p$ will assume this form, if we substitute,

for	m,	. . .	$m + 1$;
"	a,	. . .	a^2;
"	b^2,	. . .	-1;
"	n,	. . .	2;
"	p,	. . .	$-\frac{1}{2}$.

Making these substitutions in Formula **A**, we have,

$$\int \frac{x^m\,dx}{\sqrt{a^2 - x^2}} = -\frac{x^{m-1}}{m}\sqrt{a^2 - x^2} + \frac{a^2(m-1)}{m}\int \frac{x^{m-2}\,dx}{\sqrt{a^2 - x^2}};$$

so that the given binomial differential depends on,

$$\int \frac{x^{m-2}\,dx}{\sqrt{a^2 - x^2}};$$

and in a similar manner this is found to depend upon,

$$\int \frac{x^{m-4}\,dx}{\sqrt{a^2 - x^2}};$$

and so on, each operation diminishing the exponent of x by 2. If m is an even number, the integral will depend, after $\frac{m}{2}$ operations, on that of,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} \quad \text{(Art. 99)}.$$

FORMULA B.

For diminishing the exponent of the parenthesis.

167. By changing the form of the given differential binomial, we have,

$$\int x^{m-1}dx(a+bx^n)^p =$$

$$\int x^{m-1}dx(a+bx^n)^{p-1}(a+bx^n) =$$

$$a\int x^{m-1}dx(a+bx^n)^{p-1} + b\int x^{m+n-1}dx(a+bx^n)^{p-1}.$$

Applying Formula A to the second term, and observing that m is changed to $m+n$, and p to $p-1$, we have,

$$\int x^{m+n-1}dx(a+bx^n)^{p-1} =$$

$$\frac{x^m(a+bx^n)^p - am\int x^{m-1}dx(a+bx^n)^{p-1}}{b(pn+m)}.$$

Substituting this value in the last equation, we have,

$$(\mathrm{B}) \ldots\ldots\ldots\ldots \int x^{m-1}dx(a+bx^n)^p =$$

$$\frac{x^m(a+bx^n)^p + pna\int x^{m-1}dx(a+bx^n)^{p-1}}{pn+m},$$

in which the exponent of the parenthesis is diminished by 1, for each operation.

APPLICATIONS.

1. Integrate the expression $dx(a^2+x^2)^{\frac{3}{2}}$.

The differential binomial $x^{m-1}dx(a+bx^n)^p$ will assume

this form, if we make $m = 1$, $a = a^2$, $b = 1$, $n = 2$, and $p = \frac{3}{2}$.

Substituting these values in the formula, we have,

$$\int dx(a^2 + x^2)^{\frac{3}{2}} = \frac{x(a^2 + x^2)^{\frac{3}{2}} + 3a^2\int dx(a^2 + x^2)^{\frac{1}{2}}}{4}.$$

Applying the formula a second time, we have,

$$\int dx(a^2 + x^2)^{\frac{1}{2}} = \frac{x(a^2 + x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2}\int \frac{dx}{\sqrt{a^2 + x^2}}.$$

But we have found (Art. **91**),

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = l\left(x + \sqrt{a^2 + x^2}\right);$$

hence,

$$\int dx(a^2 + x^2)^{\frac{3}{2}} =$$

$$\frac{x(a^2 + x^2)^{\frac{3}{2}}}{4} + 3a^2x\frac{(a^2 + x^2)^{\frac{1}{2}}}{8} + \frac{3a^4}{8} \cdot l\left(x + \sqrt{a^2 + x^2}\right) + C.$$

2. Integrate the expression, $dx\sqrt{r^2 - x^2}$.

The first member of the equation will assume this form, if we make, $m = 1$, $a = r^2$, $b = -1$, $n = 2$, and $p = \frac{1}{2}$. Substituting these values in the formula, we have,

$$\int dx\sqrt{r^2 - x^2} = \frac{1}{2}x(r^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}r^2\int \frac{dx}{\sqrt{r^2 - x^2}};$$

whence, by substitution (Art. **99**),

$$\int dx\sqrt{r^2 - x^2} = \frac{1}{2}x(r^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}r^2 \sin^{-1}\frac{x}{r} + C.$$

FORMULA 𝕮.

For diminishing the exponent of the variable without the parenthesis, when it is negative.

168. It is evident that Formula 𝕬 will only diminish $m - 1$, the exponent of the variable, when m is positive. We are now to determine a formula for diminishing this exponent when m is negative.

From Formula 𝕬, we deduce,

$$\int x^{m-n-1}dx(a + bx^n)^p = \frac{x^{m-n}(a + bx^n)^{p+1} - b(m + np)\int x^{m-1}dx(a + bx^n)^p}{a(m - n)};$$

changing m, to $-m + n$, we have,

$$(𝕮) \ldots\ldots\ldots\ldots \int x^{-m-1}dx(a + bx^n)^p = \frac{x^{-m}(a + bx^n)^{p+1} + b(m - n - np)\int x^{-m+n-1}dx(a + bx^n)^p}{-am},$$

in which formula, it should be remembered that the negative sign has been attributed to the exponent m.

APPLICATIONS.

1. Integrate the expression $x^{-2}dx(2 - x^2)^{-\frac{3}{2}}$.

The first member of Equation (𝕮) will assume this form, if we make $m = 1$, $a = 2$, $b = -1$, $n = 2$, and $p = -\frac{3}{2}$. Substituting these values, we have,

$$\int x^{-2}dx(2 - x^2)^{-\frac{3}{2}} = -\frac{x^{-1}(2 - x^2)^{-\frac{1}{2}}}{2} + \int (2 - x^2)^{-\frac{3}{2}}dx.$$

The differential term in the second member will be integrated by the next formula.

Formula $\mathfrak{D}$.

For diminishing the exponent of the parenthesis when it is negative.

169. It is evident that Formula $\mathfrak{B}$ will only diminish p, the exponent of the parenthesis, when p is positive. We are now to determine a formula for diminishing this exponent when p is negative.

We find, from Formula $\mathfrak{B}$,

$$\int x^{m-1}dx(a+bx^n)^{p-1} =$$

$$\frac{-x^m(a+bx^n)^p + (m+np)\int x^{m-1}dx(a+bx^n)^p}{pna};$$

writing for p, $-p+1$, we have,

$$(\mathfrak{D}) \ldots \ldots \ldots \int x^{m-1}dx(a+bx^n)^{-p} =$$

$$\frac{x^m(a+bx^n)^{-p+1} - (m+n-np)\int x^{m-1}dx(a+bx^n)^{-p+1}}{na(p-1)}.$$

When $p = 1$, $p - 1 = 0$; the second member becomes infinite, and the given expression becomes a rational fraction.

APPLICATIONS.

1. Integrate the expression, $\int dx(2 - x^2)^{-\frac{3}{2}}$.

The first member of Equation **D** will assume this form, if we make $m = 1$, $a = 2$, $b = -1$, $n = 2$, and $p = -\frac{3}{2}$. Substituting these values, we have,

$$\int dx(2-x^2)^{-\frac{3}{2}} = \frac{x(2-x^2)^{-\frac{1}{2}}}{2};$$

since the coefficient of the second term, in the formula, becomes zero.

Returning, then, to the example under the last formula, we have,

$$\int x^{-2}dx(2-x^2)^{-\frac{3}{2}} = -\frac{x^{-1}(2-x^2)^{-\frac{1}{2}}}{2} + \frac{x(2-x^2)^{-\frac{1}{2}}}{2} + C.$$

2. By means of Formula **D**, we are able to integrate the expression,

$$\frac{dz}{(a^2+z^2)^p} = dz(a^2+z^2)^{-p},$$

when p is a whole number.

The general formula will assume this form, if we make $m = 1$, $x = z$, $a = a^2$, $b = 1$, $n = 2$.

Each application of the formula will reduce the exponent $-p$, by 1, until the integral will finally depend on that of

$$\frac{dz}{a^2+z^2} = \frac{1}{a}\tan^{-1}\frac{z}{a} + C \quad \text{(Art. 99)}.$$

Formula E.

When the variable enters into both terms of the binomial.

170. Let it be required to integrate the expression,

$$\frac{x^q dx}{\sqrt{2ax-x^2}} = x^q dx(2ax-x^2)^{-\frac{1}{2}}.$$

The second member may be placed under the form,

$$\int x^{q-\frac{1}{2}}dx(2a-x)^{-\frac{1}{2}}.$$

We apply Formula **A**, by making,

$$m=q+\frac{1}{2},\quad n=1,\quad p=-\frac{1}{2},\quad a=2,\quad b=-1;$$

we shall then have,

$$\int x^{q-\frac{1}{2}}dx(2a-x)^{-\frac{1}{2}} =$$

$$-\frac{x^{q-\frac{1}{2}}(2a-x)^{\frac{1}{2}}}{q}+\frac{2a(q-\frac{1}{2})}{q}\int x^{q-\frac{3}{2}}dx(2a-x)^{-\frac{1}{2}}.$$

If we observe that,

$$x^{q-\frac{1}{2}}=x^{q-1}x^{\frac{1}{2}},\quad \text{and}\quad x^{q-\frac{3}{2}}=x^{q-1}x^{-\frac{1}{2}},$$

and pass the fractional powers of x within the parentheses, we shall have,

$$(\mathbf{B})\ \ldots\ldots\ldots\ldots\quad \int\frac{x^q dx}{\sqrt{2ax-x^2}} =$$

$$-\frac{x^{q-1}\sqrt{2ax-x^2}}{q}+\frac{(2q-1)a}{q}\int\frac{x^{q-1}dx}{\sqrt{2ax-x^2}}.$$

Each application of this formula diminishes the exponent of the variable without the parenthesis by 1. If q is a positive and entire number, we shall have, after q reductions,

$$\int\frac{dx}{\sqrt{2ax-x^2}}=\text{ver-sin}^{-1}\frac{x}{a}+C\quad (\text{Art. } \mathbf{99}).$$